2

F-C

Nonlinear Scaling Laws for Parametric Receiving Arrays

Part II Numerical Analysis

prepared under:

Contract N00039-75-C-0259
ARPA Order 2910
Program Code 5G10

Authors:

J. W. Kesner
Westinghouse Electric Corporation
Oceanic Division
Annapolis, Maryland

F. H. Fenlon
Applied Research Laboratory
Pennsylvania State University
State College, Pennsylvania

June 30, 1976

Westinghouse Electric Corporation
Oceanic Division
P. O. Box 1488
Annapolis, Maryland 21404

6

Nonlinear Scaling Laws for Parametric Receiving Arrays.

Part II. Numerical Analysis,

prepared under:

15

Contract N00039-75-C-0259,
ARPA Order-2910
Program Code 5G10

Authors:

10

J. W. Kesner

Westinghouse Electric Corporation
Oceanic Division
Annapolis, Maryland

F. H. Fenlon

Applied Research Laboratory
Pennsylvania State University
State College, Pennsylvania

11 3Ø June 30, 1976

12 12Øp.

Westinghouse Electric Corporation
Oceanic Division
P. O. Box 1488
Annapolis, Maryland 21404

Nonlinear Scaling Laws for Parametric Receiving Arrays

Part II Numerical Analysis

by

J. W. Kesner and F. H. Fenlon

ABSTRACT

This report outlines the procedures used in a computer program for calculating nonlinear scaling laws for parametric receiving arrays. The basic equations, the program input and output, the program listing, and sample computer runs are included. The program output is in the form of curves for harmonic amplitudes as a function of range, the "extra dB loss" as a function of range, and parametric receiving array beam patterns at given ranges. Both detailed analytical solutions and approximate solutions are available in the computer program.

TABLE OF CONTENTS

PAGE

THEORY 1

APPROXIMATE EXPRESSIONS 4

PROGRAM INPUT 6

PROGRAM OUTPUT 8

COMMENTS ON THE PROGRAM 10

REFERENCE 12

APPENDIX A A1

FIGURES F1

PROGRAM LISTING L1

THEORY

Part I of this report covers the theoretical analysis that forms the basis of the computer program. The salient equations are presented here to indicate what the computer program does.

The spectral amplitudes Ψ_n of the axial pressure field are given by the solutions to the differential equations

$$\frac{d}{dR}(\Psi_n) = -(A+C)\Psi_n - j(B-SWCN)\Psi_n - jD\sum_{m=-\infty}^{\infty}\Psi_{n-m}\Psi_m \qquad (1)$$

where Ψ_n is the complex spectral amplitude of the n^{th} harmonic in the pressure field

R is the normalized range = range/Rayleigh distance

A = 0 for plane wave cases

= $1/R$ for spherical wave cases

= $R/(1+R^2)$ for mixed (plane and spherical wave) cases

B = $1/(1+R^2)$ for mixed cases with phase

= 0 for plane, spherical, and mixed-no phase cases

C = $n^2(f_s/f_o)^2(\alpha_o r_o)$ATTF

f_s/f_o = source frequency to pump frequency ratio

$\alpha_o r_o$ = attenuation coefficient at f_o times the Rayleigh distance

ATTF = 1 + 0.69145*SALN/(1+n^2*SWK^2)

SALN = water salinity in parts per 1000

SWK = sea water K

D = $(n/4)(f_s/f_o)\sigma_o$

σ_o = scaled source tone

SWCN = 0.69145*SALN*DDT*C_o*$(k_o r_o)$*n/(1+n^2*SWK^2)

C_o = small signal speed of sound (=1500 m/s in program)

$k_o r_o$ = wavenumber times Rayleigh distance at f_o

DDT = $0.538*10^{-9}*(1-6.54*10^{-4}*\text{PRES})$

PRES = pressure in atmospheres.

A differential equation solving routine, VODQ, which is described in Appendix A, is used to solve equation (1) for Ψ_n. Only those equations which are non-zero are included in the solution vector. The program user specifies the number (NHL) of lower harmonics (of f_s) and the number (NHH) of higher harmonics (of f_o) that he wants to consider. These harmonics are the only ones that are used in the differential equation. For example if NHL = 3, NHH = 2, and f_o/f_s = 10, then the solution vector would only contain

$$\Psi_1, \Psi_2, \Psi_3, \Psi_7, \Psi_8, \Psi_9, \Psi_{10}, \Psi_{11}, \Psi_{12}, \Psi_{13}, \Psi_{17}, \Psi_{18}, \Psi_{19}, \Psi_{20}, \Psi_{21}, \Psi_{22}, \Psi_{23}.$$

By definition $\Psi_o = 0.$

and $\Psi_{-n} = \Psi_n^*$ = complex conjugate of Ψ_n.

By specifying NHL, NHH, and f_o/f_s the infinite sum in equation (1) becomes a finite sum over at most

$$\text{NHL} + \text{NHH}*(2*\text{NHL}+1)$$

different positive values of m. For harmonics other than the ones determined by NHL, NHH, and (f_o/f_s), $\Psi_n = 0$.

The initial values of Ψ_n that are used depend on the type of problem that is being solved. If the user simply wants to see the harmonics generated by a single frequency then the initial value of $\Psi_1 = (0+_j1)$ and all other initial Ψ_n values are zero.

If the user wants an up-converstion case then

$$\Psi_{NFR} = (0+_j1) \qquad \text{where NFR} \doteq f_o/f_s$$

and $\Psi_1 = (0+_jP)$

where $P = PSO$ for a plane wave case

$= PSO*EXP(-\alpha_o r_o/NFR^2)$ otherwise.

$PSO =$ signal to pump amplitude ratio.

At each normalized range R the incoming signal is fed into the differential equation as

$$\Psi_1 = PSO*EXP(-\alpha_o r_o*R/NFR^2)*\{\sin(ARG)+_j\cos(ARG)\}$$

where $ARG = 2*(k_o r_o)*R*(f_s/f_o)*\sin^2(\Theta/2)$

and $\Theta =$ angle of intersection between signal and pump wave normals.

If the user is interested in a difference frequency case then

$$\Psi_{NFR-1} = (0+_j0.5) \quad \text{and} \quad \Psi_{NFR} = (0+_j0.5)$$

for the two primary frequencies and the difference frequency amplitude will be generated in Ψ_1.

The extra dB loss for Ψ_1 is computed from the formulas

$EXDB = -\{20*\log_{10}(\Psi_1(R)) + 8.686*\alpha_o r_o*R\}$ for plane wave cases

$EXDB = -\{20*\log_{10}(\Psi_1(R)) + 8.686*\alpha_o r_o*(R-1) + 20*\log_{10}(R)\}$ for spherical wave cases

$EXDB = -\{20*\log_{10}(\Psi_1(R)) + 8.686*\alpha_o r_o*R + 20*\log_{10}(\sqrt{1+R^2})\}$ for mixed cases.

APPROXIMATE EXPRESSIONS

Approximate expressions for $\psi_1(R)$ and $\psi_{NFR+1}(R,\theta)$ were generated in part I of this report. These are also included in the computer program as an alternative to solving the differential equations. The user has the choice of two expressions for $\psi_1(R)$. One uses the E1 function and the other involves a definite integral. The integral form seems to give better results. These expressions are given below.

Approximate expression for $\psi_1(R)$ --- E1 form.

$\psi_1(R)$ = EXP $(-\alpha_o r_o * R)$ for plane wave cases

$$\psi_1(R) = \frac{\text{EXP }(-\alpha_o r_o * R)}{RP\sqrt{1+(\sigma_o/2)^2 * \text{EXP}(4\alpha_o r_o) * \{\text{E1}(2\alpha_o r_o) - \text{E1}(2\alpha_o r_o * RP)\}^2}}$$

where RP = R for spherical wave cases

and RP = 1+R for mixed cases.

Approximate expression for $\psi_1(R)$ --- integral form.

$$\psi_1(R) = \frac{\text{EXP }(-\alpha_o r_o * R)}{RP\sqrt{1+(\sigma_o/2)^2 * \{\int_a^b \text{EXP}(-2\alpha_o r_o * R)/RP \; dR\}^2}}$$

where a = 0, b = R, RP = 1 for plane wave cases

a = 1, b = R, RP = R for spherical wave cases

a = 0, b = R, RP = $\sqrt{1+R^2}$ for mixed cases.

Approximate expression for $\psi_{NFR+1}(R,\theta)$

$\psi_{NFR+1}(R,\theta)$ =

$$\{(\sigma_o/2) * \text{EXP}(-\alpha_{NFR+1} r_o * R)/RP\} * \int_a^b XP * \psi_1(X) * \text{EXP}(\alpha_o r_o * X) * \cos(KX) \; dX \qquad (2)$$

where $\alpha_{NFR+1} r_o = (1+f_s/f_o)^2 * \alpha_o r_o$

$K = 2(k_s r_o) * \sin^2(\theta/2)$

$k_s r_o = (f_s/f_o) * k_o r_o$

$k_o r_o = (k_o a)^2/2$

$k_o a$ = wavenumber at f_o times radius of circular piston projector

$a = 0$, $b = R$, $RP = 1$, $XP = 1$ for plane wave cases

$a = 1$, $b = R$, $RP = R$, $XP = X$ for spherical wave cases

$a = 0$, $b = R$, $RP = \sqrt{1+R^2}$, $XP = \sqrt{1+X^2}$ for mixed cases

θ = angle of intersection between signal and pump wave normals.

PROGRAM INPUT

The computer program may be run from a remote terminal or by cards as a batch job. The data for the program is in free field format which means that the data does not have to be in any particular column. Input variables which begin with the letters I, J, K, L, M, N are integers and should not have any decimal point. All other variables must have a decimal point. For lines of input data that have more than one number the data values must be separated by commas.

Input 1 NOC = number of separate cases being run

Input 2 LUP, LPS, NFR, NHL, NHH, NAR, NSG, LAP, NSC, IBM

LUP = 1 for one frequency harmonic studies
= 2 for up-conversion cases
= 3 for difference frequency cases

LPS = 0 for plane wave cases
= 1 for spherical wave cases
= 2 for mixed cases with phase
= 3 for mixed cases without phase

NFR = f_o/f_s = pump frequency to signal frequency ratio

NHL = number of lower harmonics (of f_s)

NHH = number of higher harmonics (of f_o)

NAR = number of $\alpha_o r_o$ values

NSG = number of σ_o values

LAP = 1 if the approximate expressions are to be used
= 0 if the analytical solution is to be used

NSC = 0 for range scaling without σ_o
= 1 for range scaling with σ_o

IBM = 1 for beam pattern (only if LUP =2 and LAP = 1)
= 0 for no beam pattern

Input 3 XMAX = maximum range scale value

Input 4 AR(L), L = 1, ..., NAR $\alpha_o r_o$ values

Input 5 SG(L), L = 1, ..., NSG σ_o values

Input 6 SALN, PRES, TEMC, ITA, SWK, AKOA

SALN = salinity of water in parts per 1000

PRES = pressure in atmospheres

TEMC = temperature in degrees centigrade
(not used in present version of program)

ITA = 1 for Marsh & Schulkin form

= 2 for Russian form
(not used in present version of program

SWK = sea water K

AKOA = $k_o a$ = wavenumber at f_o times piston radius

Input 7 IAP (input only if LAP = 1)

IAP = 1 for E1 approximate form

= 2 for integral approximate form

Input 8 RBM, TMAX (input only if IBM = 1)

RBM = normalized range R at which beam pattern is taken

TMAX = maximum angle in degrees between pump and signal normals. The beam pattern is from 0° (on axis) to TMAX.

Input 9 PSO, LPP (input only if LUP = 2)

PSO = signal to pump amplitude ratio

LPP = 1 for only the ψ_{NFR-1} curve plotted

= 2 for only the ψ_{NFR+1} curve plotted

= 3 for both the ψ_{NFR-1} and ψ_{NFR+1} curves plotted

PROGRAM OUTPUT

The program output is in the form of curves for the axial harmonic amplitudes ($\psi_1(R)$, $\psi_2(R)$, $\psi_{NFR-1}(R)$, $\psi_{NFR+1}(R)$) as a function of range, the "extra dB loss" of $\psi_1(R)$ as a function of range, and parametric receiving array beam patterns ($\psi_{NFR+1}(R,\theta)$) at given ranges. A variety of range scales are used. For plane wave cases the range scale is $\sigma_o R$ or R. For spherical wave cases the range scale is $\sigma_o \ln(R)$ or $\ln(R)$. For mixed cases the range scale is σ_o arcsinh(R) or arcsinh(R). The "extra dB loss" curves are plotted against R. The $\psi_{NFR-1}(R)$ and $\psi_{NFR+1}(R)$ curves are plotted against R for plane, spherical, and mixed cases.

Many cases were run during the development of the program in order to test the validity of the answers that the computer program was giving. Several of these test cases are included in the figures to show the comparisons between different assumptions, such as plane wave, spherical wave, and mixed wave cases, and to show the accuracy of the approximate expressions. Several figures are included to show the versatility of the computer program since it will handle harmonic studies, up-conversion cases, and beam pattern studies. All the figures shown were also chosen with the idea of demonstrating how the harmonic amplitudes and beam patterns change with the various parameters, such as $\alpha_o r_o$, σ_o, and f_o/f_s.

Figures 1 and 2 are plane wave cases that show $\psi_1(R)$, $\psi_2(R)$, and $\psi_3(R)$ for two values of $\alpha_o r_o$ and four values of $\sigma_o/\alpha_o r_o$.

Figures 3 and 4 are spherical wave cases that show $\psi_1(R)$, $\psi_2(R)$, and extra dB loss for two values of $\alpha_o r_o$.

Figures 5, 6, and 7 are mixed wave cases that show $\psi_1(R)$, $\psi_2(R)$, and extra dB loss for three values of $\alpha_o r_o$ and compare the curves generated by the approximate and analytical methods. In addition, figure 7 covers a range of five σ_o values.

Figure 8 compares the spherical and mixed wave curves for ψ_2/ψ_1 and ψ_3/ψ_1 with similar curves previously generated by Fenlon[1].

Figure 9 shows the similarity of the two up-converted signals $\psi_{NFR-1}(R)$ and $\psi_{NFR+1}(R)$.

Figure 10 compares $\psi_{NFR+1}(R)$ as generated with the mixed phase and mixed no phase assumptions and with the mixed approximate integral expressions.

Figure 11 shows $\psi_{NFR+1}(R)$ for three values of f_o/f_s.

Figure 12 shows $\psi_{NFR+1}(R)$ for two extreme values of $\alpha_o r_o$ and two extreme values of σ_o.

Figure 13 shows up-converted beam patterns for three values of f_o/f_s at two different ranges. One range is at the peak of the $\psi_{NFR+1}(R)$ curve and the other range is in the far-field of the $\psi_{NFR+1}(R)$ curve.

Figure 14 shows up-converted beam patterns for four value of σ_o at two different ranges. Some of these patterns are near-field patterns and some are far-field patterns because the peak of the $\psi_{NFR+1}(R)$ curve changes with σ_o.

COMMENTS ON THE PROGRAM

The user should have a good idea of the normalized range scale he is interested in so that he can input an appropriate value for XMAX. It was found that using NSC = 0 (no σ_o scale) was often more convenient when plotting curves for a range of σ_o values. For up-conversion cases using the approximate expressions it is important to use a value of XMAX that is not too large because the $\psi_1(R)$ curve is precomputed at a finite number of points over the range up to XMAX and interpolation is used for values of R other than those prestored. Because of the steep rise and peak of the $\psi_1(R)$ curve it is necessary to have several points in this region. Otherwise, a poor representation of $\psi_1(R)$ will result and the resulting $\psi_{NFR+1}(R)$ curve will be no good. Program running cost was the main reason for prestoring the $\psi_1(R)$ curve.

One suggested change to the program is to include a multiplying factor G_m inside the summation of equation (1), but only for the m = - ∞ to m = 0 terms in the mixed with phase case. The suggested G_m term is

$$G_m = 1/(1 + 2m/((1+R^2)*(f_o/f_s))).$$

Another suggestion for making the program more complete is to replace the term cos(KR) in equation (2) with the term sin(KR) + j cos(KR).

If the product KR in equation (2) is large enough then false lobes can appear because of the accuracy limitations in numerical integration of a rapidly varying function. These false lobes have shown up in figure 13 where the product KR is very large for some of the cases.

As an added feature, the computer program has been set up so that difference frequency cases can also be run. Most of the necessary changes to the program have been made, but have not been thoroughly checked out. If anyone would like to use the difference frequency part of the program it is suggested that he contact the authors first.

For more details of exactly what the computer program does, the reader is referred to the program listing which is included in this report.

REFERENCE

1. F.H. Fenlon, "A Recursive Procedure for Computing the Nonlinear Spectral Interactions of Progressive Finite-Amplitude Waves in Nondispersive Fluids", J. Acoust. Soc. Am. 50, 1299-1312 (1971).

APPENDIX A

Included in this appendix are descriptions of the packaged routines that are used in the program. The routines VODQ, ROMBS, and DEI are from the JPL Fortran V Subroutine Directory, Edition No. 4, October 1970. The routines TSCALE, TSETUP, and TPLOT are from the Westinghouse Research Laboratories Fortran library.

12.1. INITIAL VALUE PROBLEM

12.1.1. NUMERICAL SOLUTION, S.P. AND D.P.

12.1.1.1. IDENTIFICATION

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS.

LANG	FILE	ELT/VERS	SIZE	ENTRY NAMES
F-V	LIB*JPL$	VOIGO/JPL	5257(8)=2735(10)	VODQ,VODQ1,VODQG

SUBROUTINE USED: #NONE#

COGNIZANT PERSON: F. T. KROGH, SECTION 314, JPL, 1969 JUNE 10

12.1.1.2. PURPOSE

THIS SUBROUTINE COMPUTES THE NUMERICAL SOLUTION OF THE INITIAL VALUE PROBLEM FOR A SYSTEM OF ONE OR MORE ORDINARY DIFFERENTIAL EQUATIONS.

12.1.1.3. REFERENCE

1. F. T. KROGH, VODQ/SVDQ/DVDQ - VARIABLE ORDER INTEGRATORS FOR THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS. SECTION 314 SUBROUTINE WRITE-UP, JPL, MAY 1969.

2. F. T. KROGH, 'ON TESTING A SUBROUTINE FOR THE NUMERICAL INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS', JPL, SECTION 314, TM NO. 217, MAY 1969.

12.1.1.4. PRECISION

THIS SUBROUTINE IS PRIMARILY SINGLE PRECISION, USING DOUBLE PRECISION ARITHMETIC ONLY AT A FEW CRITICAL POINTS. TWO OTHER VERSIONS OF THIS SUBROUTINE ARE ALSO AVAILABLE: SVDQ, WHICH IS ENTIRELY SINGLE PRECISION, AND DVDQ, WHICH IS ENTIRELY DOUBLE PRECISION. SEE THEIR WRITE-UPS FOR MINOR DIFFERENCES IN USAGE.

12.1.1.5. REMARKS

THE ORDINARY DIFFERENTIAL EQUATIONS MAY BE OF ORDERS 1,2,3, OR 4, AND NEED NOT ALL BE OF THE SAME ORDER. IT IS SUGGESTED THAT IF AN

ORDINARY DIFFERENTIAL EQUATION IS OF ORDER 2,3, OR 4, IT SHOULD BE TREATED AS SUCH, RATHER THAN BEING REFORMULATED AS A SET OF 2,3, OR 4 FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS. THIS PERMITS THE USE OF INTEGRATION FORMULAS WHICH HAVE BETTER NUMERICAL STABILITY.

12.1.1.6. METHOD

THIS SUBROUTINE USES LINEAR MULTISTEP PREDICTOR-CORRECTOR FORMULAS OF THE ADAMS-MOULTON TYPE WITH THE APPROPRIATELY GENERALIZED FORMULAS FOR ORDINARY DIFFERENTIAL EQUATIONS OF ORDERS 2,3, AND 4. THE SUBROUTINE SELECTS THE ORDER (1,2,..., OR 9) OF THE FORMULAS INDEPENDENTLY FOR EACH ORDINARY DIFFERENTIAL EQUATION OF THE SYSTEM AND THEN SELECTS A STEP SIZE TO BE USED FOR THE ENTIRE SYSTEM. THESE SELECTIONS OF ORDER AND STEPSIZE ARE RECONSIDERED AND POSSIBLY CHANGED AT EACH STEP. THE SELECTION IS BASED UPON MAINTAINING NUMERICAL STABILITY AND MEETING THE USER'S REQUESTED LOCAL ACCURACY.

THE SUBROUTINE PROVIDES SPECIAL RETURNS BASED UPON EITHER THE NUMBER OF STEPS, THE VALUE OF THE INDEPENDENT VARIABLE, VALUES OF THE DEPENDENT VARIABLES, OR VALUES OF AUXILIARY FUNCTIONS. ONLY THE FIRST TWO OF THESE FEATURES ARE DESCRIBED IN THIS WRITE-UP. SEE REFERENCE 1 FOR THE OTHER TWO FEATURES WHICH INVOLVE USE OF THE ENTRY POINT VODQG.

12.1.1.7. USAGE

```
      INTEGER             NEQ,KD(#N1#),IFLAG,MXSTEP,KSTEP,KEMAX,
                          KQ(#N7#)
      REAL                EP(#N2#),HMINA,HMAXA,EMAX
      REAL                F(#N3#),DT(10,#N4#)
      DOUBLE PRECISION    T,Y(#N5#),H,DELT,TFINAL,YN(#N6#)


      CALL   VODQ  (NEQ,T,Y,F,KD,EP,IFLAG,H,HMINA,HMAXA,DELT,TFINAL,
             MXSTEP,KSTEP,KEMAX,EMAX,KQ,YN,DT)
             GO TO 6


    4 CALL   VODQ1 6 GO TO  (10,10,30,40,50,60,70,80),IFLAG

   10 [COMPUTE  F( )  AS  A FUNCTION OF  T  AND  Y( ).  IF
      IFLAG = 2  THE VALUE OF  T  WILL BE UNCHANGED FROM THE PRE-
      VIOUS RETURN.  THUS THE USER CAN EFFECT SOME REDUCTION OF EX-
      ECUTION TIME BY EVALUATING ANY SUBEXPRESSIONS OF  F( )  WHICH
      DEPEND ON  T  BUT NOT ON  Y( )  ONLY WHEN  IFLAG = 1, AND
```

```
       REUSING THESE VALUES WITHOUT RECOMPUTING THEM WHEN
       IFLAG = 2.]
       GO TO 4


    30 [AN OUTPUT POINT HAS BEEN REACHED UNDER CONTROL OF  DELT.]
       GO TO 4


    40 [  T  HAS REACHED THE VALUE  TFINAL.]
       GO TO ...


    50 [AN OUTPUT POINT HAS BEEN REACHED UNDER CONTROL OF
       MXSTEP.]
       GO TO 4


    60 [ACCURACY APPARENTLY BEING LIMITED BY ROUND-OFF ERROR.  SUG-
       GEST THAT  EP( )  BE INCREASED.]
       GO TO ...


    70 [ ABS(H) NEEDS TO BE LESS THAN  HMINA  TO ACHIEVE THE
       ACCURACY SPECIFIED BY  EP( ).]
       GO TO ...


    80 [ERRONEOUS OR INCOMPATIBLE SETTING OF CONTROL PARAMETERS.]
```

THE DIMENSIONING PARAMETERS MUST SATISFY:

```
#N1# /.GE. 1  IF  KD(1)>0,
     \.GE. NEQ  IF  KD(1)<0

#N2# /.GE. 1  IF  EP(1)>0,
     \.GE. #K#  IF  EP(1)<0,  WHERE  #K#  IS THE SMALLEST INTEGER
     FOR WHICH  EP(#K#) .GE. 0

#N3#,#N4#,#N7# .GE. NEQ

#N5#,#N6#   .GE. #KDTOTAL#

#KDTOTAL#  /.EQ. KD(1)*NEQ  IF  KD(1)>0
           \.EQ. SUM OF  IABS(KD(I))  FOR  I = 1,...,NEQ  IF
           KD(1)<0
```

BEFORE CALLING VODQ, THE USER MUST ASSIGN INITIAL VALUES TO

T AND Y( ) AND SET THE PARAMETERS NEQ,KD( ),EP( ),H,HMINA, HMAXA,DELT,TFINAL, AND MXSTEP. THE USER MAY CHANGE THE VALUES OF EP( ) WHENEVER IFLAG = 1, AND MAY CHANGE THE VALUES OF HMINA,HMAXA,DELT,TFINAL, AND MXSTEP AT ANY TIME.

THE USER MUST COMPUTE F( ) WHENEVER THE SUBROUTINE RETURNS WITH IFLAG = 1 OR 2.

THE SUBROUTINE WILL CHANGE THE VALUES OF T,Y( ),IFLAG,H,KSTEP, KEMAX,EMAX,KQ( ),YN( ), AND DT( ) DURING THE INTEGRATION. ON RETURNS WITH IFLAG = 3,4, OR 5 THE CURRENT VALUE OF THE SOLUTION WILL BE CONTAINED IN T,Y( ), AND F( ).

THE SUBROUTINE PARAMETERS ARE DEFINED AS FOLLOWS:

NEQ NUMBER OF DIFFERENTIAL EQUATIONS IN THE SYSTEM.

T INDEPENDENT VARIABLE. MUST BE SET TO ITS INITIAL VALUE BY THE USER. THEREAFTER T WILL BE CHANGED BY THE SUBROUTINE.

(Y(I),I=1,#KDTOTAL#) DEPENDENT VARIABLES. FOR SYSTEM OF FIRST ORDER EQUATIONS Y(I) IS THE I**#TH# DEPENDENT VARIABLE. IN THE CASE OF HIGHER ORDER EQUATIONS, THE FIRST LOCATIONS OF Y CONTAIN THE DEPENDENT VARIABLE ASSOCIATED WITH THE FIRST EQUATION AND ITS DERIVATIVES UP TO ORDER ONE LESS THAN THE ORDER OF THE EQUATION (WITH THE LOWER ORDER DERIVATIVES STORED FIRST), THEN FOLLOW THE VARIABLES ASSOCIATED WITH THE SECOND EQUATION, ETC. FOR EXAMPLE, FOR THE SYSTEM:

U'' = F1(U,U',V,V',T)
V'' = F2(U,U',V,V',T)

THE Y( ) ARRAY WOULD BE USED AS FOLLOWS:

Y(1) = U
Y(2) = U'
Y(3) = V
Y(4) = V'

(F(I),I=1,NEQ) DERIVATIVE VALUES. IF #DI# DENOTES THE ORDER OF THE I**#TH# EQUATION, THEN F(I) IS THE #DI#**#TH# DERIVATIVE OF THE I**#TH# COMPONENT WITH RESPECT TO T. THE USER MUST PROVIDE THE CODE TO COMPUTE F( ), GIVEN T

AND Y( ).

KD( ) ORDER OF THE DIFFERENTIAL EQUATIONS IN SYSTEM. DIFFERENTIAL EQUATIONS WITH ORDER GREATER THAN 4 CAN ONLY BE INTEGRATED BY BREAKING THEM INTO LOWER ORDER EQUATIONS, OR BY CHANGING CERTAIN DATA STATEMENTS IN THE INTEGRATOR. IF ALL OF THE DIFFERENTIAL EQUATIONS ARE OF THE SAME ORDER THEN IT SUFFICES TO SET KD(1) EQUAL TO THAT ORDER (KD(1)>0). IF DIFFERENTIAL EQUATIONS OF MIXED ORDER ARE TO BE INTEGRATED, KD IS A VECTOR WITH KD(1)<0 TO INFORM THE INTEGRATOR THAT THIS IS THE CASE. THE ORDER OF THE I**#TH# EQUATION IS THEN GIVEN BY ABS(KD(I)). (FOR I>1 , KD(I) MAY BE EITHER POSITIVE OR NEGATIVE).

EP( ) PARAMETER USED TO CONTROL THE LOCAL ERROR. IF ONE WANTS THE SAME LOCAL ERROR BOUND ON ALL COMPONENTS, THEN EP(1)>0 AND THE ESTIMATE OF THE LOCAL ERROR IN EACH COMPONENT IS KEPT LESS THAN EP(1)/10. FOR DIFFERENT ERROR BOUNDS ON DIFFERENT COMPONENTS, LET EP(1)<0 FOR I<#K#, AND LET EP(#K#) .GE. 0. THE LOCAL ERROR CONTROL FOR THE I**#TH# COMPONENT IS THEN BASED ON ABS(EP(I)) FOR I<#K# AND ON EP(#K#) FOR I .GE. #K#. SUGGESTIONS ON HOW TO SELECT EP( ) ARE GIVEN IN REFERENCE 1.

IFLAG PARAMETER USED FOR COMMUNICATION BETWEEN THE INTEGRATOR AND THE USER. THE INTEGRATOR SETS IFLAG AS FOLLOWS:

=1 THE VALUE OF Y( ) FOR THE CURRENT STEP HAS BEEN PREDICTED. THE USER SHOULD COMPUTE F( ) AND CALL VODQ1. IF A RELATIVE ERROR TEST IS DESIRED, THE NEW VALUE(S) OF EP SHOULD ALSO BE COMPUTED HERE.

=2 THE VALUE OF Y( ) FOR THE CURRENT STEP HAS BEEN CORRECTED. THE USER SHOULD COMPUTE F( ) AND CALL VODQ1.

=3 AN OUTPUT POINT HAS BEEN REACHED (SEE USAGE OF DELT), TO CONTINUE THE INTEGRATION CALL VODQ1.

=4 T=TFINAL. IF VODQ1 IS CALLED WITH T=TFINAL AND IFLAG=4, IFLAG IS SET EQUAL TO 8. IF THE VALUE OF TFINAL IS CHANGED THE INTEGRATION WILL CONTINUE.

=5 KSTEP=KSOUT (SEE THE DESCRIPTION OF MXSTEP).

=6 EMAX>0.1 AND IT APPEARS THAT REDUCING H WILL NOT HELP REDUCE THE GLOBAL ERROR BECAUSE OF ROUND-OFF ERROR. IF

THIS OCCURS, A LARGER VALUE OF EP( ) (OR OF ABS(EP(KEMAX)) IF EP IS A VECTOR) SHOULD PROBABLY BE USED. IF EP IS NOT INCREASED, TOO SMALL A STEPSIZE IS LIABLE TO BE USED. WE HAVE FOUND THAT REPLACING EP( ) WITH 32*EMAX*EP( ) WORKS REASONABLY WELL. (NOTE THAT EMAX IN MOST CASES WILL BE ONLY SLIGHTLY LARGER THAN 0.1).

=7 ABS(H)<HMINA. THIS MAY BE THE RESULT OF HALVING H, OF COMING TO THE END OF THE STARTING PHASE WITH ABS(H)<HMINA, OR OF THE USER INCREASING THE VALUE OF HMINA. IF ONE WISHES TO CONTINUE WITH THE CURRENT VALUE OF H, SET HMINA .LE. H AND CALL VODQ1. IF THE STEPSIZE HAS JUST BEEN HALVED (IN WHICH CASE EMAX>0.1), ONE MAY CONTINUE WITH THE OLD STEPSIZE BY SIMPLY CALLING VODQ1. (SUCH AN ACTION IS RISKY WITHOUT A CAREFUL ANALYSIS OF THE SITUATION.) IF THE STEPSIZE HAS NOT JUST BEEN HALVED, SIMPLY CALLING VODQ1 WILL RESULT IN THE INTEGRATION CONTINUING WITH THE CURRENT VALUE OF H, AND A RETURN TO THE USER WITH IFLAG=7 WILL OCCUR AT THE END OF EVERY STEP UNTIL ABS(H) .GE. HMINA.

=8 ERROR INDICATION. THE CONDITIONS WHICH MAY CAUSE IFLAG TO BE SET EQUAL TO 8 ARE LISTED BELOW.

WHEN CALLING VODQ:

NEQ .LE. 0;
H*DELT .LE. 0;
KD=0 OR KD>4 (KD A SCALAR);
KD(#I#)=0 OR ABS(KD(#I#))>4 FOR SOME #I#,#I#=1,2,..., NEQ (KD A VECTOR);

WHEN CALLING VODQ1:

IFLAG=4 AND T=TFINAL;
EP=0 AND HMAXA≠0;
H*(TFINAL-(INITIAL VALUE OF T))<0 AND THE ESTIMATED ERROR IN EXTRAPOLATING TO TFINAL FROM THE INITIAL POINT (USING A FIRST ORDER METHOD) IS LARGER THAN PERMITTED BY EP;

WHEN CALLING VODQG WITH NG≠0:

NSTOP<0 OR NSTOP>ABS(NG) (SEE USAGE OF THE VODQG ENTRY IN THE COMPLETE WRITE-UP)

IF VODQ1 IS CALLED WITH IFLAG SET EQUAL TO 8, THE PROGRAM IS STOPPED AND AN ERROR MESSAGE IS PRINTED.

H THE STEPSIZE. H CAN BE EITHER POSITIVE OR NEGATIVE. WHEN SELECTING THE INITIAL VALUE OF H, THE USER SHOULD REMEMBER THE FOLLOWING:

1. THE INTEGRATOR IS CAPABLE OF CHANGING H QUITE RAPIDLY AND THUS THE INITIAL CHOICE IS NOT CRITICAL.

2. IF IT DOES NOT LEAD TO PROBLEMS IN COMPUTING THE DERIVATIVES (E.G. BECAUSE OF OVERFLOW OR BECAUSE OF TRYING TO COMPUTE THE SQUARE ROOT OF A NEGATIVE NUMBER), IT IS BETTER TO CHOOSE H MUCH TOO LARGE THAN MUCH TOO SMALL.

HMINA MINIMUM STEPSIZE PERMITTED (AFTER THE INTEGRATION IS STARTED). AFTER THE INTEGRATION IS STARTED, AND WHEN H IS HALVED, ABS(H) IS COMPARED WITH HMINA. IF ABS(H)<HMINA CONTROL IS RETURNED TO THE USER WITH IFLAG=7.

HMAXA MAXIMUM STEPSIZE PERMITTED. THE STEPSIZE IS NOT DOUBLED IF DOING SO WOULD RESULT IN ABS(H)>HMAXA.

DELT OUTPUT INCREMENT. DELT MUST HAVE THE SAME SIGN AS H. INITIALLY TOUT IS SET EQUAL TO THE INITIAL VALUE OF T. WHENEVER T=TOUT A RETURN IS MADE TO THE USER WITH IFLAG=3. (THUS, IN PARTICULAR, A RETURN IS ALWAYS MADE AT THE INITIAL POINT). WHENEVER VODQ1 IS CALLED WITH IFLAG=3, TOUT IS REPLACED WITH T+DELT. IF TOUT DOES NOT FALL ON AN INTEGRATION STEP, OUTPUT VALUES ARE OBTAINED BY INTERPOLATION ON THE FIRST STEP THAT (T-TOUT)*H>0. INTERPOLATED VALUES FOR BOTH Y AND F ARE COMPUTED.

TFINAL FINAL VALUE OF T. WHEN T REACHES TFINAL, CONTROL IS RETURNED TO THE USER WITH IFLAG=4. OUTPUT VALUES ARE OBTAINED BY EXTRAPOLATION IF TFINAL DOES NOT FALL ON AN INTEGRATION STEP. IF ONE CHANGES TFINAL DURING THE INTEGRATION SO THAT (TFINAL-T)*H<0, AN IMMEDIATE INTERPOLATION IS PERFORMED TO OBTAIN THE OUTPUT VALUES.

MXSTEP MAXIMUM NUMBER OF STEPS BETWEEN OUTPUT POINTS. WHEN

VODQ1 IS CALLED WITH 3 .LE. IFLAG .LE. 5, KSOUT IS SET EQUAL TO KSTEP + MXSTEP. AT THE END OF EACH STEP, KSTEP IS INCREMENTED AND COMPARED WITH KSOUT. IF KSTEP .GE. KSOUT CONTROL IS RETURNED TO THE USER WITH IFLAG=5. THUS IF DELT IS SUFFICIENTLY LARGE, CONTROL WILL BE RETURNED TO THE USER WITH IFLAG=5 EVERY MXSTEP STEPS.

KSTEP NUMBER OF INTEGRATION STEPS TAKEN.

KEMAX INDEX OF THE COMPONENT RESPONSIBLE FOR EMAX (SEE BELOW)

EMAX LARGEST VALUE IN ANY COMPONENT OF (ESTIMATED ERROR)/#E# WHERE #E#=ABS(EP) FOR THE COMPONENT UNDER CONSIDERATION. ORDINARILY THE STEPSIZE IS HALVED IF EMAX>0.1. HOWEVER, WITH A RECENT HISTORY OF ROUND-OFF ERROR LIMITING THE PRECISION, VALUES OF EMAX AS LARGE AS 1 ARE PERMITTED.

(KQ(I),I=1,NEQ) VECTOR USED TO STORE INTEGRATION ORDERS. KQ(I) GIVES THE INTEGRATION ORDER USED ON THE I**#TH# EQUATION

(YN(I),I=1,#KDTOTAL#) WORKING SPACE. (VECTOR USED TO STORE Y AT THE END OF EACH INTEGRATION STEP).

((DT(I,J),I=1,10),J=1,NEQ) WORKING SPACE. (DIFFERENCE TABLES COMPUTED BY THE SUBROUTINE).

11.1. ONE-DIMENSIONAL

11.1.1. QUADRATURE, ONE-DIMENSION, S.P.

11.1.1.1. IDENTIFICATION

QUADRATURE, ONE-DIMENSIONAL, SINGLE PRECISION

LANG	FILE	ELT/VERS	SIZE	ENTRY NAMES
F-V	LIB*JPL$	ROMBS/JPL	1251(8)=681(10)	ROMBS,ROM2

SUBROUTINES USED: #NONE#

COGNIZANT PERSONS: W. R. BUNTON AND M. DIETHELM,
JPL, SECTION 314, 1969 SEPT 30

11.1.1.2. PURPOSE

OBTAIN APPROXIMATE EVALUATION OF A ONE-DIMENSIONAL DEFINITE INTEGRAL BY NUMERICAL QUADRATURE.

ANS = THE INTEGRAL FROM A TO B OF F(X)*DX

11.1.1.3. REFERENCES:

FOR A COMPLETE DESCRIPTION OF THIS SUBROUTINE, INCLUDING A DISCUSSION OF A VARIETY OF TEST CASES, SEE:

WILEY R. BUNTON, MICHAEL DIETHELM, AND KAREN HAIGLER, 'ROMBERG QUADRATURE SUBROUTINE FOR SINGLE AND MULTIPLE INTEGRALS', JPL

W. BUNTON, M. DIETHELM, G. WILLJE, 'MODIFIED ROMBERG QUADRATURE: A SUBROUTINE TO SUPPORT GENERAL SCIENTIFIC COMPUTING', JPL INTERNAL MEMORANDUM TM 314-208, APRIL 1, 1970.

W. BUNTON, M. DIETHELM, 'MODIFICATIONS TO THE JPL ROMBERG SUBROUTINES', TM 314-247, 1 SEPT. 1970.

11.1.1.4. METHOD

THIS SUBROUTINE COMBINES TECHNIQUES FROM 'ROMBERG' AND 'ADAPTIVE STEP' QUADRATURE METHODS. THE SUBROUTINE INITIALLY PICKS A SUBINTERVAL [A,B1] OF THE TOTAL INTERVAL [A,B] AND ATTEMPTS TO APPROXIMATE THE INTEGRAL OVER THIS SUBINTERVAL BY USING THREE

STAGES OF ROMBERG QUADRATURE. THIS REQUIRES EVALUATION OF THE INTEGRAND AT FIVE EQUALLY SPACED POINTS.

IF THIS QUADRATURE IS REGARDED AS SUCCESSFUL (SEE: ERROR CONTROL) THE SUBROUTINE WILL ADD THE VALUE OBTAINED TO A RUNNING SUM AND PROCEED TO TREAT A NEW DISJOINT SUBINTERVAL OF THE SAME OR GREATER LENGTH.

IF THIS QUADRATURE IS NOT REGARDED AS SUCCESSFUL AND THE CURRENT STEP LENGTH IS GREATER THAN HMIN THEN THE SUBROUTINE WILL REJECT THE RESULT FOR THE CURRENT SUBINTERVAL AND TAKE THE LEFT HALF OF THE SUBINTERVAL AS THE NEW SUBINTERVAL TO BE TREATED.

IF THE CURRENT STEP LENGTH IS HMIN OR SMALLER, THEN THE SUBROUTINE WILL ACCEPT THE CURRENT RESULT AND PROCEED TO THE NEXT SUBINTERVAL, WRITING A MESSAGE ON FORTRAN UNIT 6 IDENTIFYING THE SUBINTERVAL ON WHICH THE ACCURACY TEST WAS NOT SATISFIED.

11.1.1.5. ERROR CONTROL

THE QUADRATURE IS REGARDED AS SATISFACTORY OVER A PARTICULAR SUBINTERVAL IF

1. THE ESTIMATED RELATIVE ERROR OF THE QUADRATURE OVER THAT SUBINTERVAL IS AT MOST ERMAX, OR

2. THE ESTIMATED ERROR IN THAT SUBINTERVAL RELATIVE TO THE ACCUMULATED VALUE OF THE INTEGRAL UP TO AND INCLUDING THAT SUBINTERVAL IS AT MOST ERMAX.

3. THE ESTIMATED ABSOLUTE ERROR OVER THAT SUBINTERVAL IS AT MOST .1*ABS(ERMAX).

11.1.1.6. PARAMETER CHECKING

THE SUBROUTINE WILL TERMINATE EXECUTION WITH A PRINTED MESSAGE AND A STANDARD EXEC 8 WALK-BACK (RETURN 0) IF THE GIVEN PARAMETERS DO NOT SATISFY

```
A<B, OR A>B, BUT NOT A=B
0.<HMIN.LE.HSTAR.LE.HMAX AND

RELATIVE - 0.<ERMAX<1.0, BUT NOT ERMAX=0.
ABSOLUTE - ANY NEGATIVE NUMBER, BUT NOT =0.
```

IF A.GT.B OR IF HSTAR.GE.(B-A)/4, ONLY A WARNING MESSAGE IS PRINTED.

11.1.1.7. USAGE

```
      REAL      A, B, X, FOFX, HSTAR, HMIN, HMAX, ERMAX, ANS
      INTEGER   K, KEY

      [ASSIGN VALUES TO A, B, HSTAR, HMIN, HMAX, ERMAX, AND KEY]

      CALL  ROMBS(A,B,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,K,KEY)

   10 [EVALUATE THE INTEGRAND USING THE CURRENT VALUE OF  X  AND
      STORE THE RESULT IN  FOFX.]

      CALL  ROM2

      IF(K.EQ.1) GO TO 10
      [QUADRATURE IS COMPLETED.  RESULT IS IN  ANS]
```

THE SUBROUTINE PARAMETERS ARE DEFINED AS FOLLOWS:

A,B LIMITS OF INTEGRATION. REQUIRE A NOT=B.

X VARIABLE SET BY THE SUBROUTINE FOR INTEGRAND EVALUATION IN THE USER'S PROGRAM.

FOFX VALUE OF INTEGRAND COMPUTED BY USER'S PROGRAM USING THE ARGUMENT X.

HSTAR SUGGESTED INITIAL STEP SIZE. THE INITIAL STEP SIZE, H, WILL BE SET AT H=.01*(B-A) IF HSTAR.GE.(B-A)/4. OR AT H=HSTAR IF HSTAR.LT.(B-A)/4. THE FIRST SUBINTERVAL WILL BE OF LENGTH 4.*H AND WILL REQUIRE FIVE EVALUATIONS OF THE INTEGRAND AT X=A,A+H,...A+4*H. SUGGEST HSTAR.GE.(B-A)/4.

REQUIRE HMIN .LE. HSTAR .LE. HMAX.

HMIN MINIMUM ALLOWABLE STEP SIZE. REQUIRE 0. < HMIN .LE. HSTAR

HMAX MAXIMUM ALLOWABLE STEP SIZE. REQUIRE HSTAR .LE. HMAX.

ERMAX TOLERANCE ON RELATIVE OF ABSOLUTE ERROR. SEE DISCUSSION ABOVE UNDER 'ERROR CONTROL'. REASONABLE SETTINGS FOR ERMAX WOULD BE IN THE RANGE FROM 1.E-4 TO 1.E-7. IF GREATER ACCURACY IS REQUIRED, SEE THE WRITE-UP ON ROMBD. IT IS REQUIRED THAT 0.LT.ERMX.LT.1. FOR THE RELATIVE ERROR TEST. ONE SHOULD KNOW THE RANGE OF THE ANSWER

BEFORE HE USES ABSOLUTE ERROR.

ANS THE FINAL VALUE OF THE INTEGRAL. AVAILABLE WHEN ROM2 RETURNS WITH K=2.

K BRANCHING FLAG SET BY THE SUBROUTINE FOR USE IN THE USER'S PROGRAM. K=1 MEANS THE USER SHOULD EVALUATE THE INTEGRAND AT X, STORE THE VALUE IN FOFX, AND RE-ENTER ROM2. K=2 MEANS THE COMPUTATION IS COMPLETED AND THE VALUE IS IN ANS.

KEY FLAG TO CONTROL PRINTING OF ERROR MESSAGES. PREVIOUSLY, ANY VALUE OF KEY NOT = 7 WOULD WRITE A DIAGNOSTIC MESSAGE WHEN H BECAME.LT.HMIN. THE INPUT VALUES HAVE BEEN CHANGED SO THAT WHEN

ACTION

KEY=5 PRINT INTERMEDIATE T AND Y VALUES; PRINT THE HMIN DIAGNOSTIC IF DETECTED.

=6 PRINT INTERMEDIATE T AND Y VALUES; DO NOT PRINT THE HMIN DIAGNOSTIC.

=7 DO NOT PRINT THE T AND Y VALUES OR HMIN DIAGNOSTIC.

=ANY OTHER VALUE PRINT THE HMIN DIAGNOSTIC IF DETECTED.

THE T VALUES PRINTED ARE T(1,0), T(1,1), AND T(2,0). THE PRINTED Y VALUES ARE THE FUNCTIONAL EVALUATIONS Y(1) THRU Y(5) AT THE POINTS X,X+H,...,X+4H. SEE REFERENCES.

11.1.1.8. REMARKS:

THE FOLLOWING NOTES MAY BE APPLICABLE FOR DIFFICULT INTEGRANDS.

1. IF IN DOUBT, ONE SHOULD USE A SMALL VALUE OF HSTAR. THE STEP SIZE H CAN DOUBLE QUICKLY AND THE USER IS PENALIZED ONLY A SMALL NUMBER ON FUNCTIONAL EVALUATIONS WHILE HE INCREASES HIS CHANCES OF GETTING AN ACCURATE APPROXIMATION MANY FOLD.

2. BE CAUTIOUS WHEN RELATIVE ERMAX IS .GT. 10-5. IF HSTAR.LT.(B-A)/4, BUT NOT SMALL ENOUGH, AND THE FUNCTION IS OCCILATORY, VERY DIFFICULT, ETC., ROMBS CAN RETURN A WRONG ANSWER. EXAMPLE: F(X)=X*SIN30X*COSX ON THE INTERVAL (0,2PI), HSTAR=1.57, TRUE ANSWER=-.20967248, A RELATIVE TOLERANCE OF .1 GAVE -.25E-5, AND A RELATIVE TOLERANCE OF .01 GAVE 4.188; BAD ANSWERS.

3. ALSO FOR RELATIVE ERMAX .LT.10**-6. IF THE RELATIVE ERROR IS ASKING FAR GREATER THAN 6 SIGNIFICANT DIGITS ONE IS PUSHING THE ACCURACY OF THE UNIVAC 1108. ON A HIGHLY OCCILATORY, VERY DIFFICULT PROBLEM, ROMBS MAY BE TAKING THOUSANDS MORE FUNCTIONAL EVALUATIONS AND NOT REACHING THE ACCURACY IT DID AT 10**-6. EXAMPLE: SAME F(X) AS ABOVE WHEN GIVEN TOLERANCE WAS 10**-6, ANS=.20967380 WITH A 2009 FUNCTIONAL EVAUATIONS; BUT WHEN WHEN ERMAX=10**-8, ANS=.20967765 AND 10,267 FUNCTIONAL EVALUATIONS.

4. IF ONE WANTS A ROUGH APPROXIMATION OF THE INTRGRAL, HE CAN SET HMIN=HSTAR=HMAX. A FIXED STEP INTRGRATION OF THE FUNCTION WILL TAKE PLACE. BE SURE THAT KEY=7, OR MANY DIAGNOSTICS MAY BE PRINTED.

4.2.13. EXPONENTIAL INTEGRAL

4.2.13.1. IDENTIFICATION

EXPONENTIAL INTEGRAL

LANG	FILE	ELT/VERS	SIZE	ENTRY NAMES
F-V	LIB*JPL$	DEI/JPL		DEI

SUBROUTINES USED: DEXP,DLOG

COGNIZANT PERSON: E.W. NG, JPL, SECTION 314, 1970, SEPT. 14

4.2.13.2. PURPOSE

COMPUTE THE EXPONENTIAL INTEGRAL IN DOUBLE PRECISION ARITHMETIC. FOR X .GT. 0, THE EXPONENTIAL INTEGRAL, EI, IS DEFINED AS

EI(X) = INTEGRAL FROM T=-INFINITY TO T=X OF
(EXP(T)/T)*DT

WHERE THE INTEGRAL IS TO BE INTERPRETED AS THE CAUCHY PRINCIPAL VALUE. FOR X .LT. 0, EI(X) =-E1(-X), WHERE

E1(Z)=INTEGRAL FROM T=Z TO T=INFINITY OF
(EXP(-T)/T)*DT

4.2.13.3. METHOD

THIS SUBROUTINE COMPUTES THE EXPONENTIAL INTEGRAL BY CHEBYSHEV RATIONAL APPROXIMATIONS FROM W.J. CODY AND H.C. THACHER, JR., MATH. COMP. VOL. 22, PP. 641-650, AND VOL. 23, PP. 289-303.

4.2.13.4. ACCURACY

EXTENSIVE TESTS WERE PERFORMED ON THE UNIVAC 1108 AND THE FOLLOWING ACCURACY STATISTICS WERE FOUND:

INTERVAL OF X	MAXIMUM RELATIVE	RMS RELATIVE
(-150, -4)	2.2D-16	5.1D-17

(-4, -1)	7.5D-17	1.2D-17
(-1, 0)	8.7D-18	1.4D-18
(0, 0.5)	1.8D-16*	3.2D-17*
(0.5, 6)	5.5D-17	1.0D-17
(6, 12)	1.6D-17	3.0D-18
(12, 24)	2.9D-17	7.1D-18
(24, 100)	8.9D-17	1.9D-17

CF. E.W. NG, COMM. ACM VOL. 13, #7, PP. 448-449.

4.2.13.5. USAGE

DOUBLE PRECISION DEI, X

USE THE FOLLOWING FUNCTION IN A FORTRAN ARITHMETIC STATEMENT:

DEI(X)

WHERE DEI IS THE COMPUTED VALUE OF THE EXPONENTIAL INTEGRAL.

4.2.13.6. LIMITATIONS

EI(0)=-INFINITY IS APPROXIMATED BY -7.2D75 AND EI(X .GT. 174.673) IS APPROXIMATED BY +7.2D75.

THESE LIMITATIONS WERE ORIGINALLY IMPOSED FOR THE IBM/360 AND HAVE NOT BEEN MODIFIED FOR THE UNIVAC 1108.

4.2.13.7. ERROR EXITS

NONE

TSCALE

Specifications:

```
      SUBROUTINE TSCALE (V,NPTS,VLOW,VHIGH,FIRST,IXORY)
      INTEGER NPTS,IXORY
      REAL V(NPTS),VLOW,VHIGH
      LOGICAL FIRST
```

Purpose:

To scan the array V to obtain the minimum and maximum values, which are then used to establish a range which produces a rational scale for the axis.

Usage:

The subroutine should be called as

```
          TSCALE(V,NPTS,VLOW,VHIGH,FIRST,IXORY)
```

where

V = array containing data to be scanned

NPTS = number of points in V to be scanned

VLOW,HIGH = receives upper and lower values for the range of V

FIRST = .TRUE. for first call for each plot,
.FALSE. for subsequent calls which may be made for a multiple curve plot. Note that all curves should be scanned via TSCALE for a multiple curve plot.

IXORY = 0 for abscissa,1 for ordinate

TSCALE (Cont'd)

Note:

The abscissa is eight grids down the page, and the ordinate is 5 grids across. TSCALE need not be used if the range of values is known.

TSETUP

Specifications:

```
     SUBROUTINE TSETUP(XLOW,XHIGH,YLOW,YHIGH,IEORF,IGRID,LABELX,LABELY)
     INTEGER IEORF,IGRID,LABELX(1),LABELY(1)
     REAL XLOW,XHIGH,YLOW,YHIGH
```

Purpose:

To set up work area before data values to be plotted are stored (via TPLOT). Labelling information, ranges of values for the abscissa and ordinate variables, and options for grid and annotation are input parameters.

Usage:

The subroutine should be called as

```
          TSETUP(XLOW,XHIGH,YLOW,YHIGH,IEORF,IGRID,LABELX,LABELY)
```

where

XLOW,XHIGH (YLOW,YHIGH) = lower and upper values for the range of the abscissa (ordinate) variable

IEORF = option to set the format for annotating the axes; supply 1HE for E-format (1PE10.3), or 1HF for F-format (F10.3).

IGRID = option to select grid; supply 1HG for grid, 1Hb (blank) for no grid.

TSETUP (Cont'd)

LABELX(LABELY) = array containing the information to label the X(Y) axis. LABELX(1) (LABELY(1)) contains the number of characters, and the labelling data itself is stored six characters per word starting in LABELX(2) (LABELY(2)), with a maximum of 30 (24) characters.

TPLOT

Specifications:

```
     SUBROUTINE TPLOT(PLOT,ICHAR,NPTS,XV,YV)
     INTEGER ICHAR,NPTS
     REAL XV(NPTS), YV(NPTS)
     LOGICAL PLOT
```

Purpose:

To store plot data and initiate plotting via PLOT control.

Usage:

The subroutine should be called as

```
          TPLOT (PLOT,ICHAR,NPTS,XV,YV)
```

where

PLOT = .FALSE. if data values are to be stored and plotting deferred (for multiple curves on one plot); .TRUE. if data values are to be stored and plotted.

ICHAR = single character used as plot symbol (e.g., 1H*)

NPTS = number of data points to be plotted

XV,YV = arrays containing the coordinate values to be plotted, i.e., (XV(I)),YV(I)) is the Ith data point.

Restrictions:

1. If more than one data point occupies the same plot position, the most recent value processed will be used.
2. Data values outside range specified by TSETUP will not be plotted, but will be listed below the plot.

Figure 1

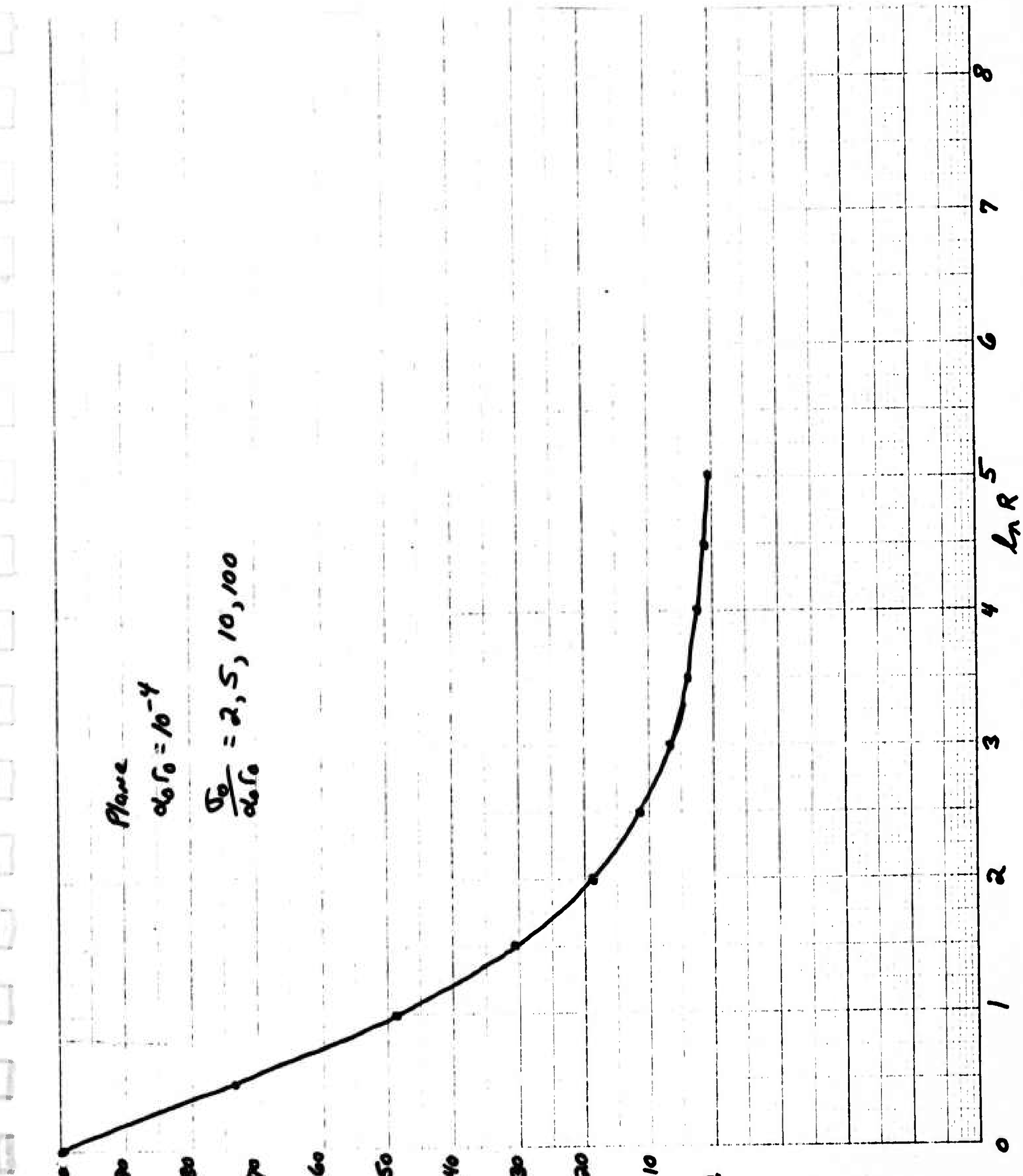
(pages F2 to F4)

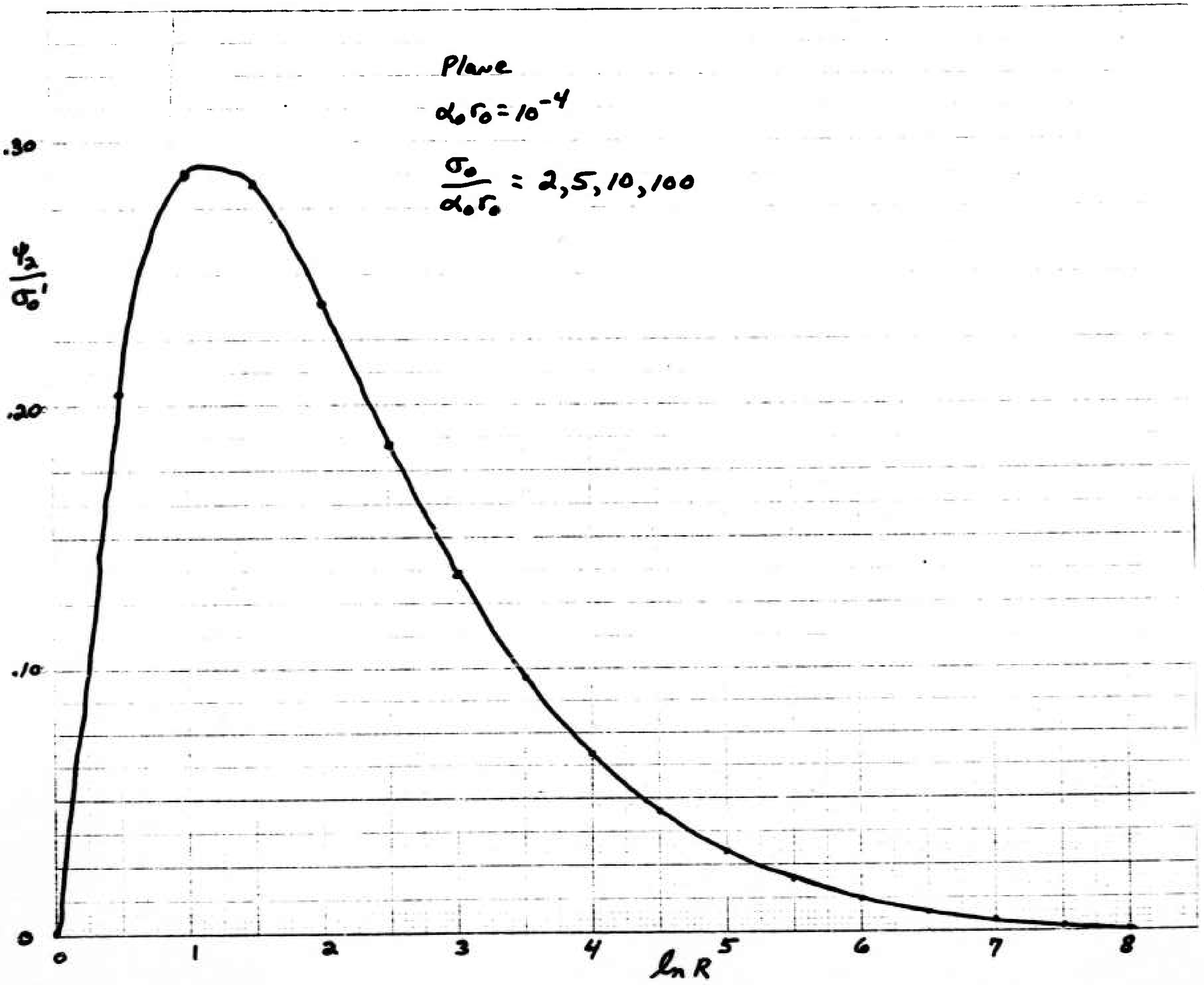
Plane wave case

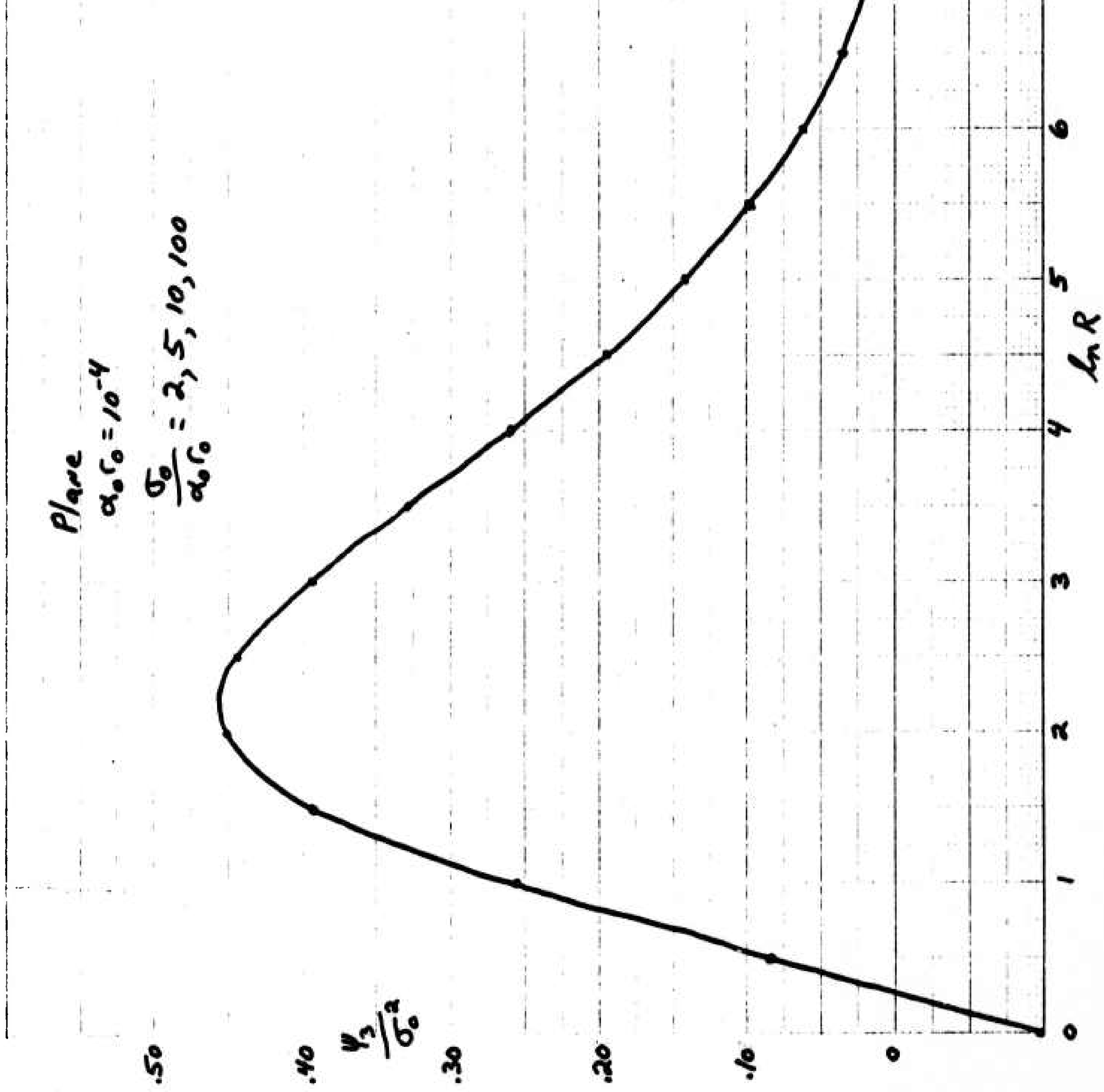
$\alpha_o r_o = 10^{-4}$

$\sigma_o/\alpha_o r_o = 2, 5, 10, 100$

Ψ_1, Ψ_2, Ψ_3


Plane
$\alpha_0\Gamma_0 = 10^{-4}$
$\frac{\sigma_0}{\alpha_0\Gamma_0} = 2, 5, 10, 100$
$\psi = \frac{\Psi}{\sigma_0}$
1.0
.90
.80
.70
.60
.50
.40
.30
.20
.10
0
$\ln R$
0
1
2
3
4
5
6
7
8


Plane
$\alpha_0 r_0 = 10^{-4}$
$\frac{\sigma_0}{\alpha_0 r_0} = 2, 5, 10, 100$
$\frac{\psi_2}{\sigma_0'}$
.30
.20
.10
0
0
1
2
3
4
5
6
7
8
$\ln R$


Plane
$\alpha_0 r_0 = 10^{-4}$
$\frac{\sigma_0}{\alpha_0 r_0} = 2, 5, 10, 100$
ψ_3/σ_0^2
.50
.40
.30
.20
.10
0
0
1
2
3
4
5
6
7
8
ln R

F4

Figure 2

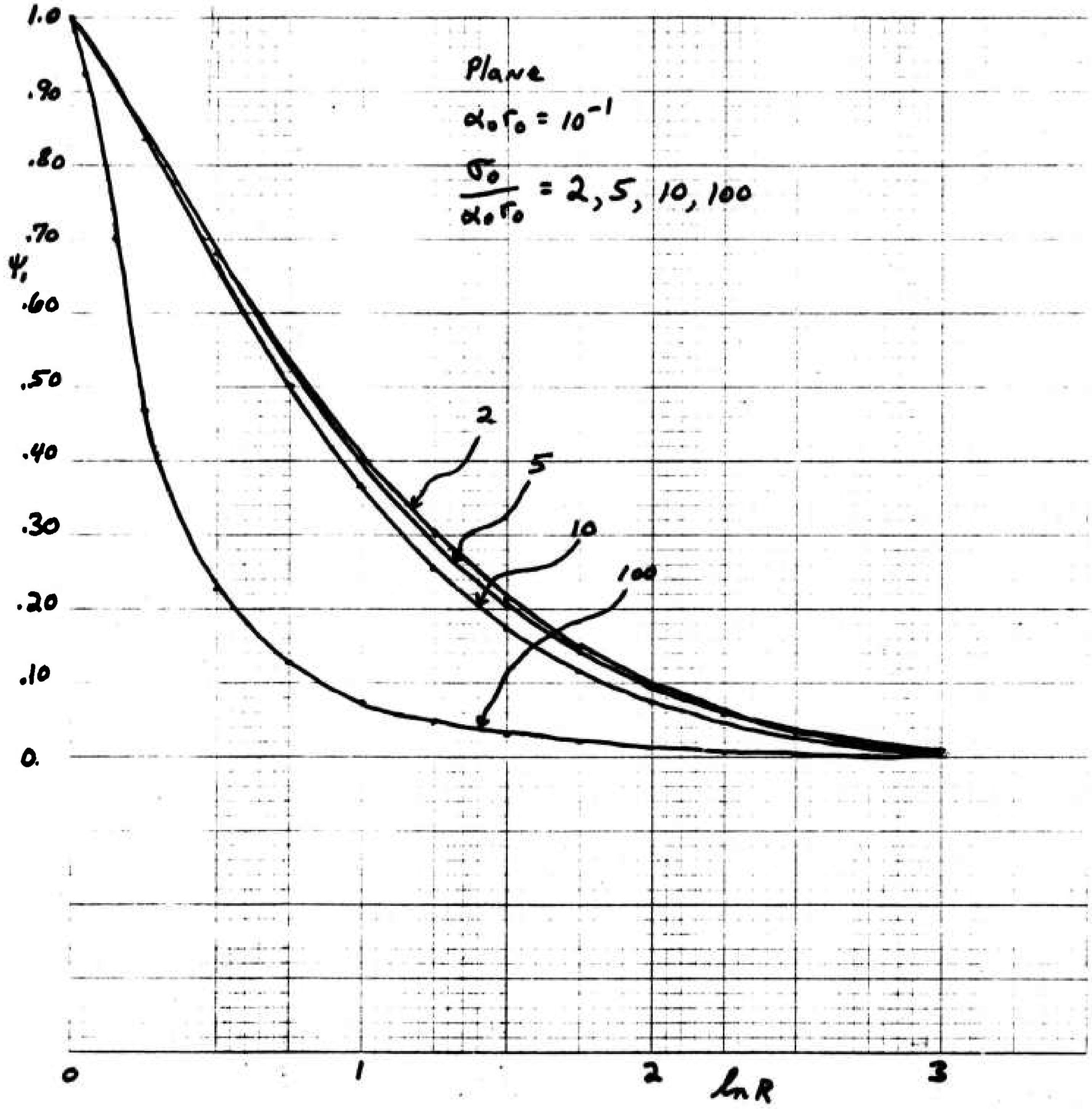
(pages F6 to F8)

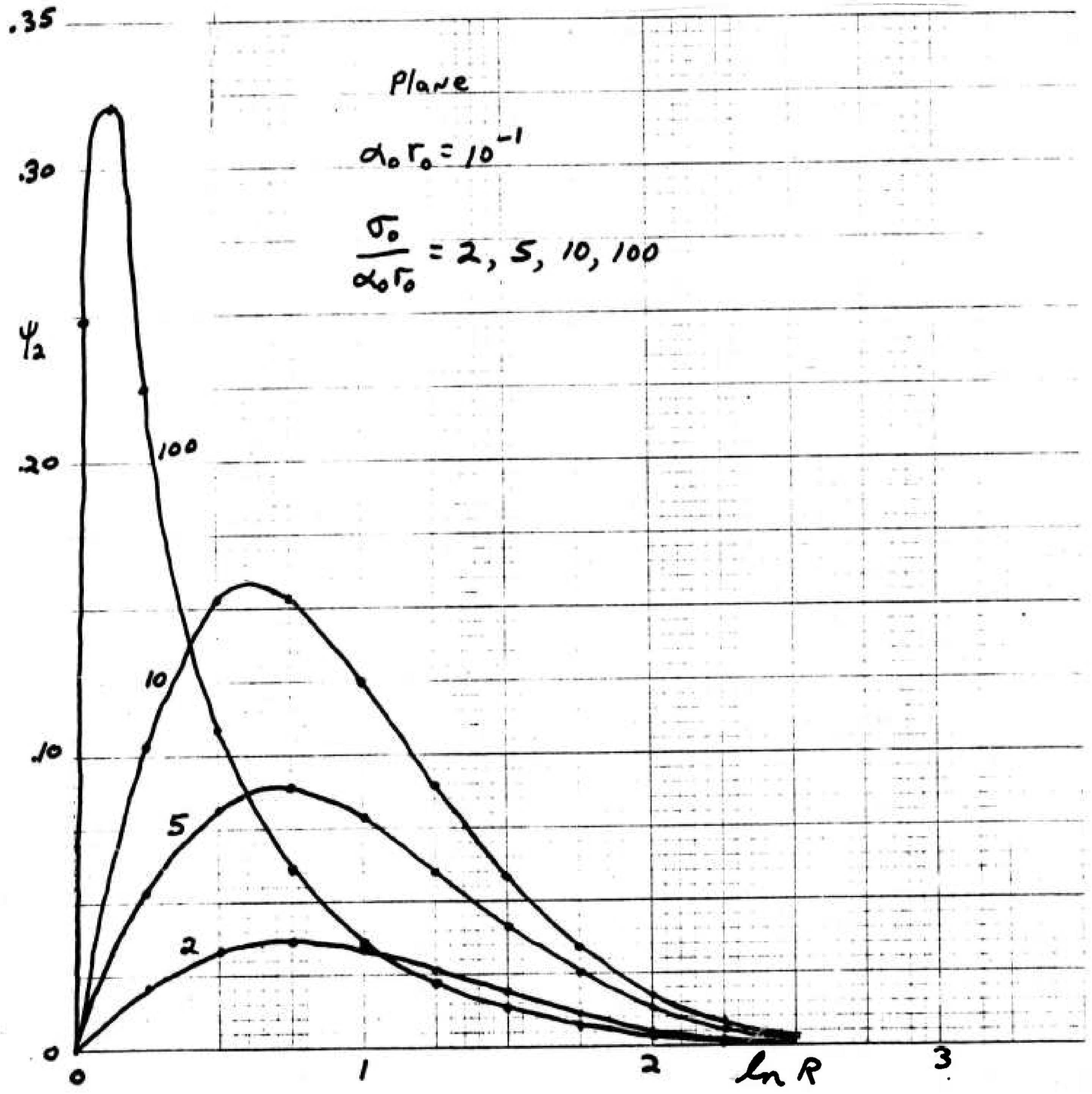
Plane wave case

$\alpha_o r_o = 10^{-1}$

$\sigma_o/\alpha_o r_o = 2, 5, 10, 100$

ψ_1, ψ_2, ψ_3


Plane
$\alpha_0 r_0 = 10^{-1}$
$\frac{\sigma_0}{\alpha_0 r_0} = 2, 5, 10, 100$
1.0
.90
.80
.70
Ψ_1
.60
.50
.40
.30
.20
.10
0.
2
5
10
100
0
1
2
3
$\ln R$


Plane
$\alpha_0 r_0 = 10^{-1}$
$\frac{\sigma_0}{\alpha_0 r_0} = 2, 5, 10, 100$
.35
.30
ψ_2
.20
.10
0
100
10
5
2
0
1
2
ln R
3

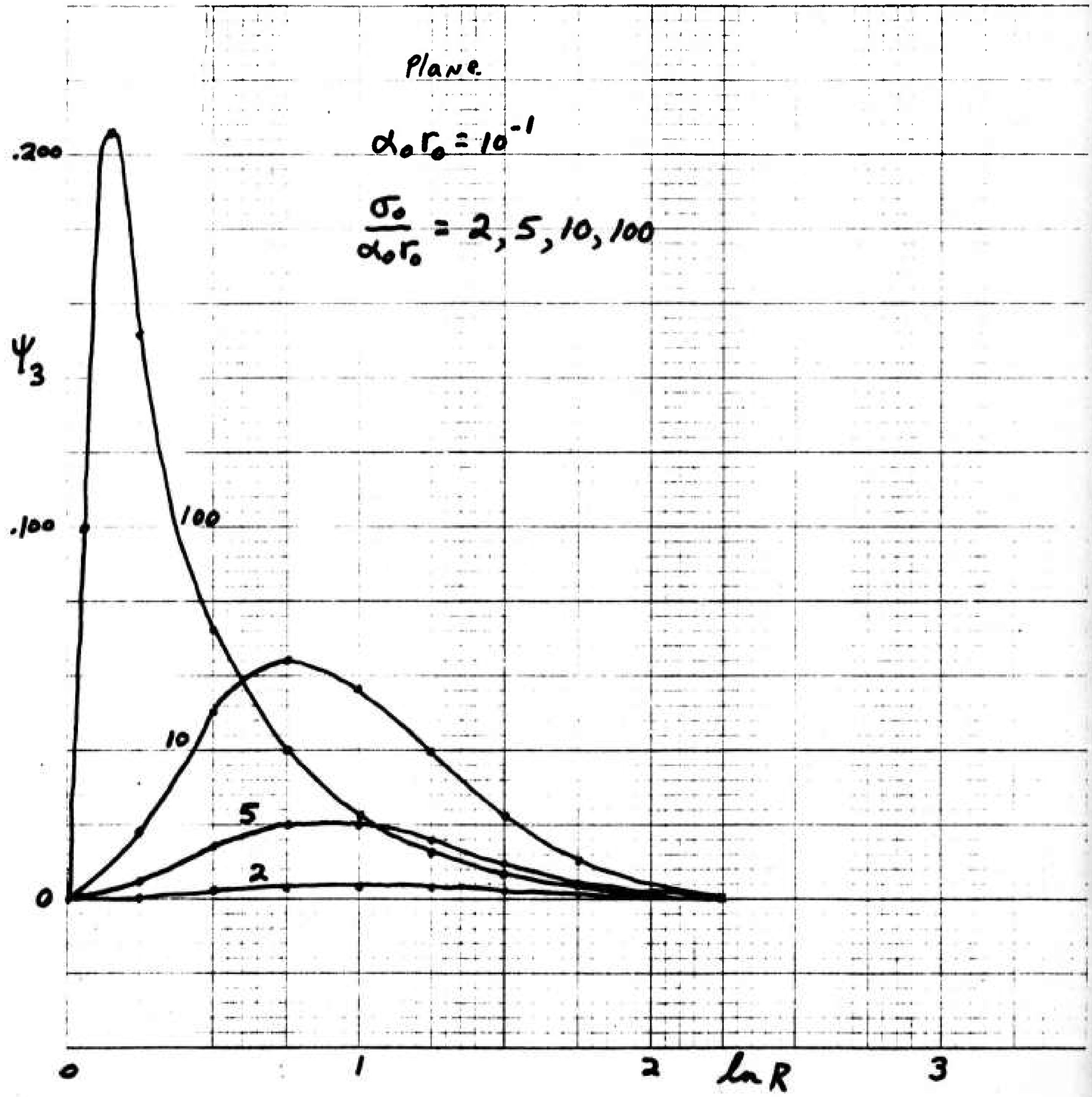

Plane
$\alpha_o r_o = 10^{-1}$
$\frac{\sigma_o}{\alpha_o r_o} = 2, 5, 10, 100$
Ψ_3
.200
.100
0
100
10
5
2
0
1
2
ln R
3

Figure 3

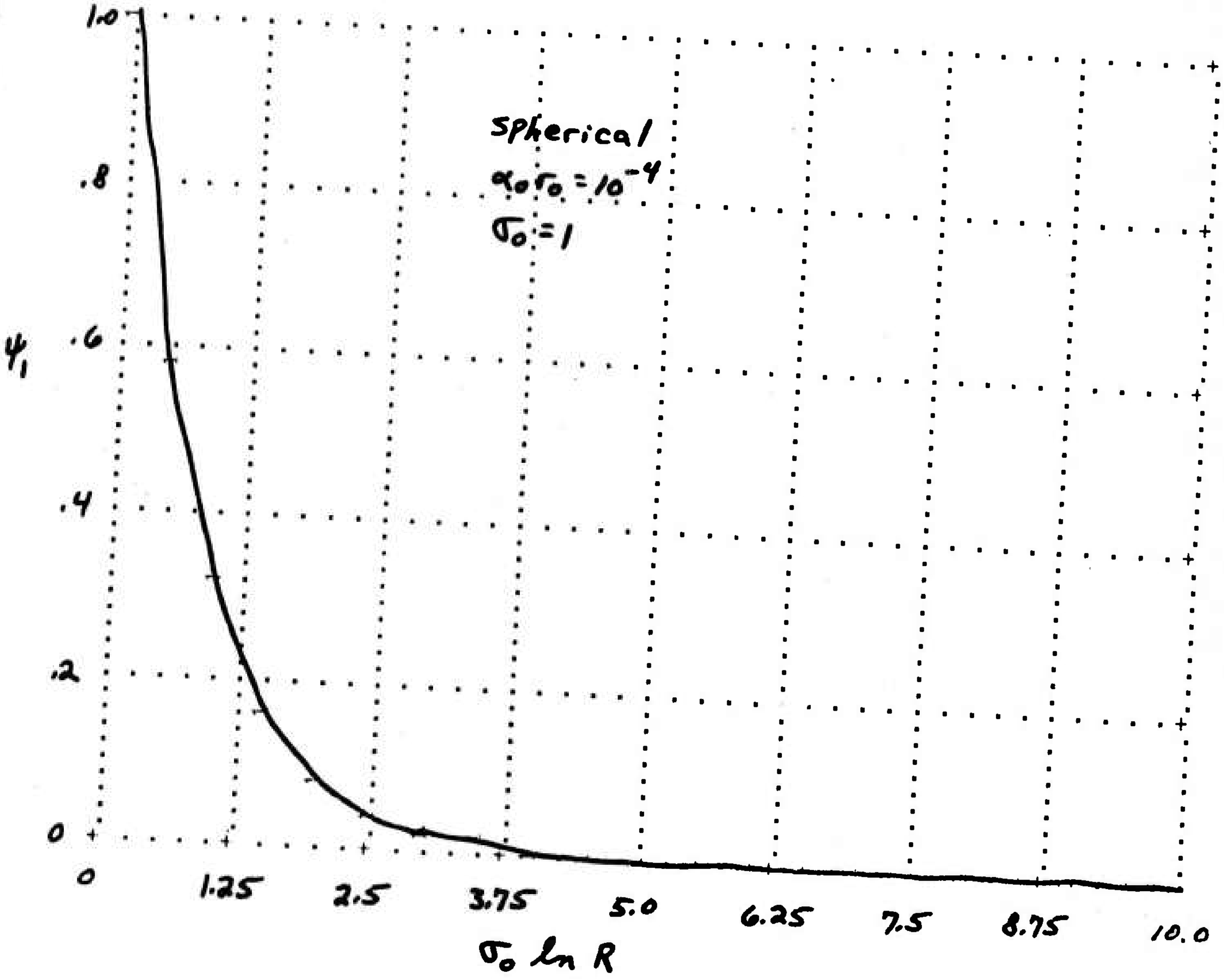
(pages F10to F12)

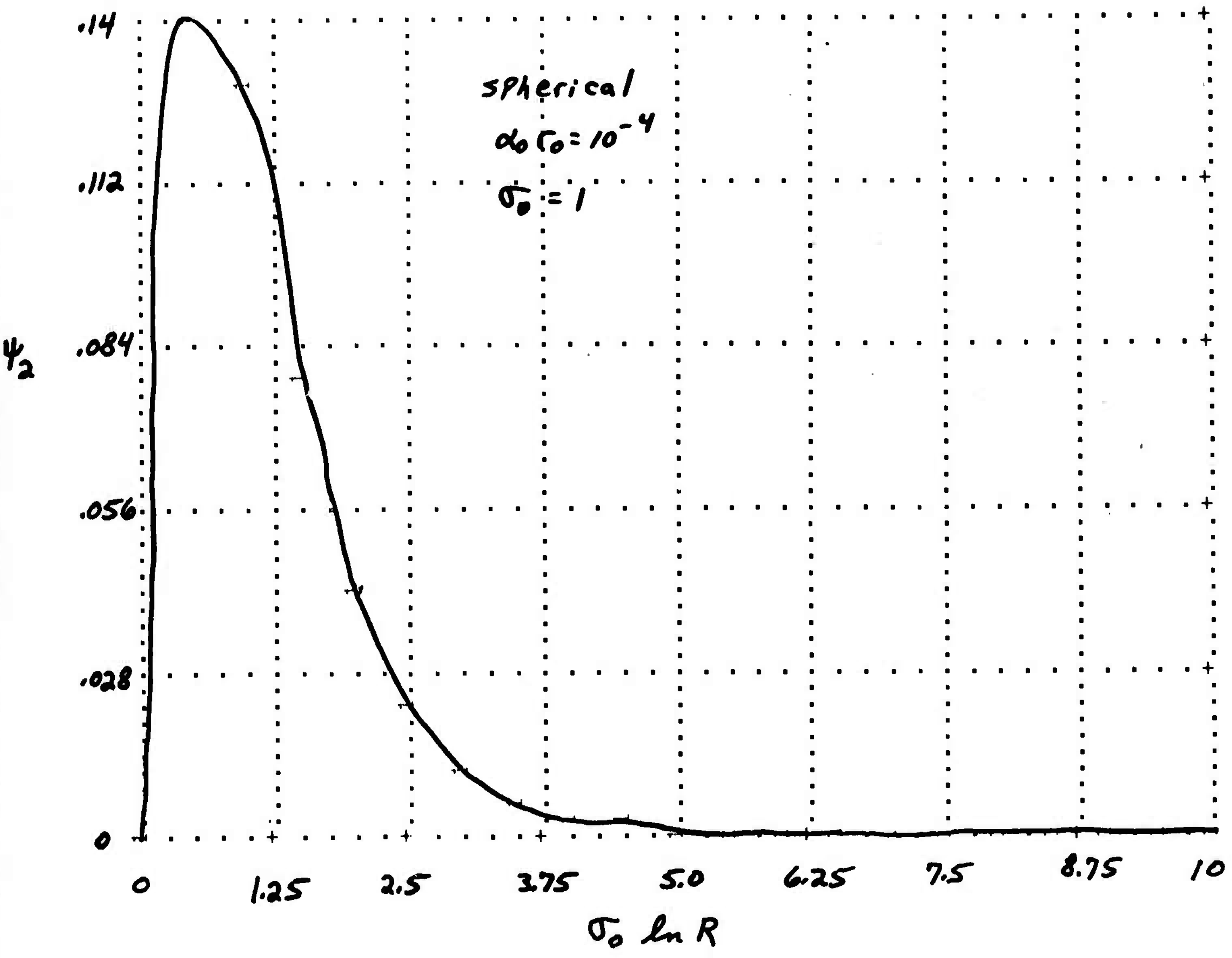
Spherical wave case

$\alpha_o r_o = 10^{-4}$

$\sigma_o = 1$

ψ_1, ψ_2, EXDB


1.0
.8
.6
.4
.2
0
ψ_1
0
1.25
2.5
3.75
5.0
6.25
7.5
8.75
10.0
$\sigma_0 \ln R$
Spherical
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 1$


.14
.112
.084
.056
.028
0
ψ_2
spherical
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 1$
0
1.25
2.5
3.75
5.0
6.25
7.5
8.75
10
$\sigma_0 \ln R$

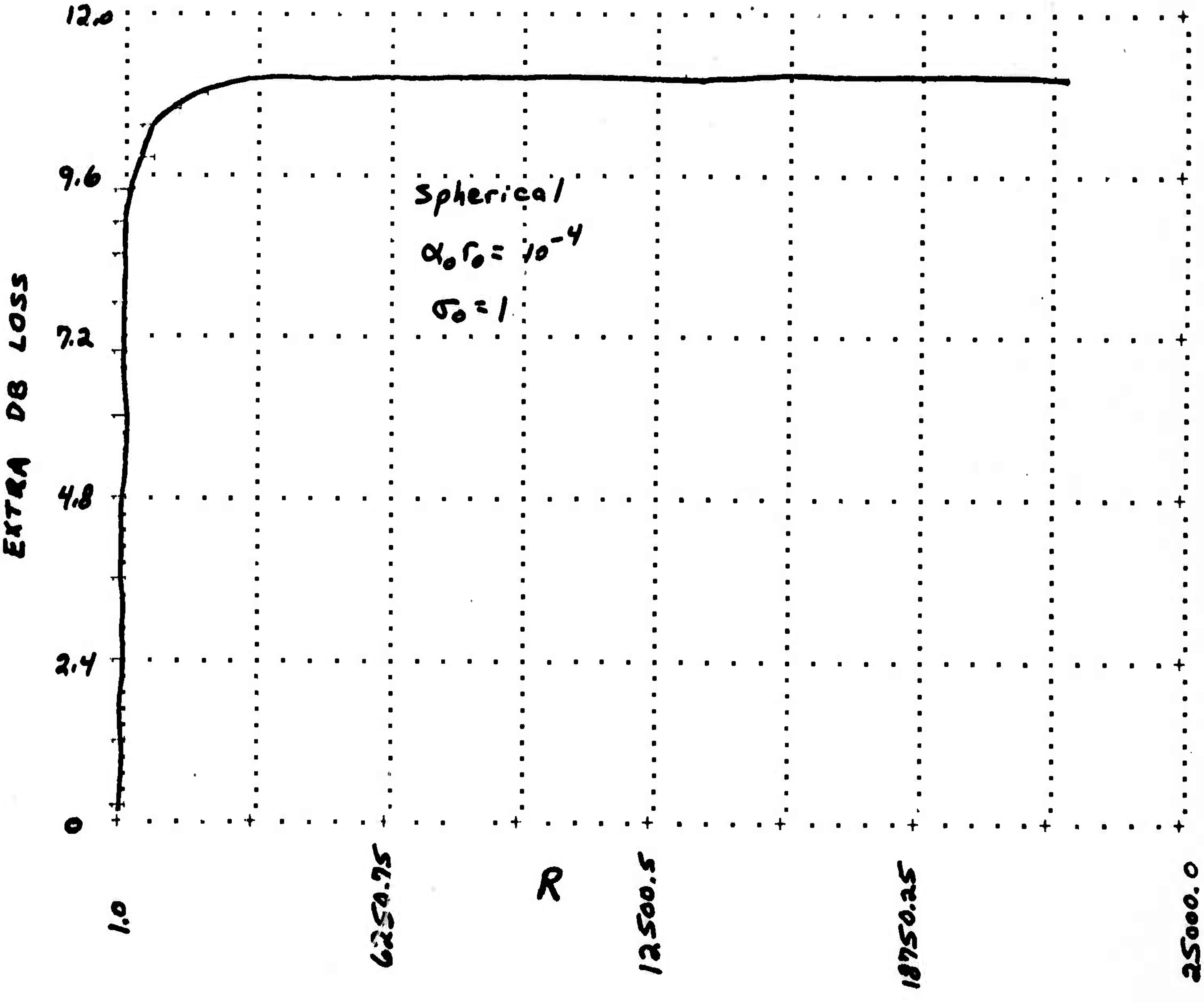

12.0
9.6
7.2
4.8
2.4
0
EXTRA DB LOSS
1.0
6250.75
R
12500.5
18750.25
25000.0
Spherical
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 1$

Figure 4

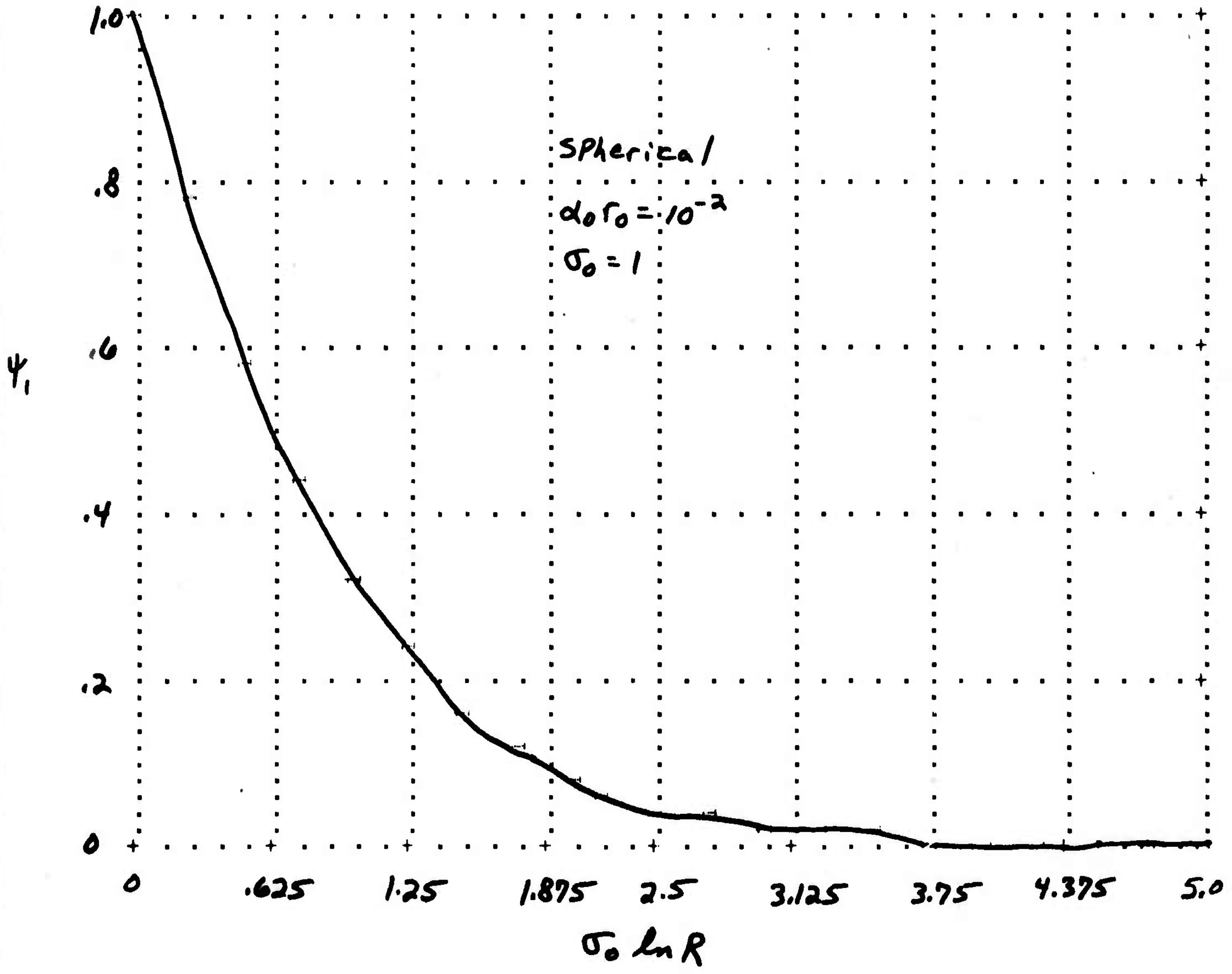
(pages F14to F16)

Spherical wave case

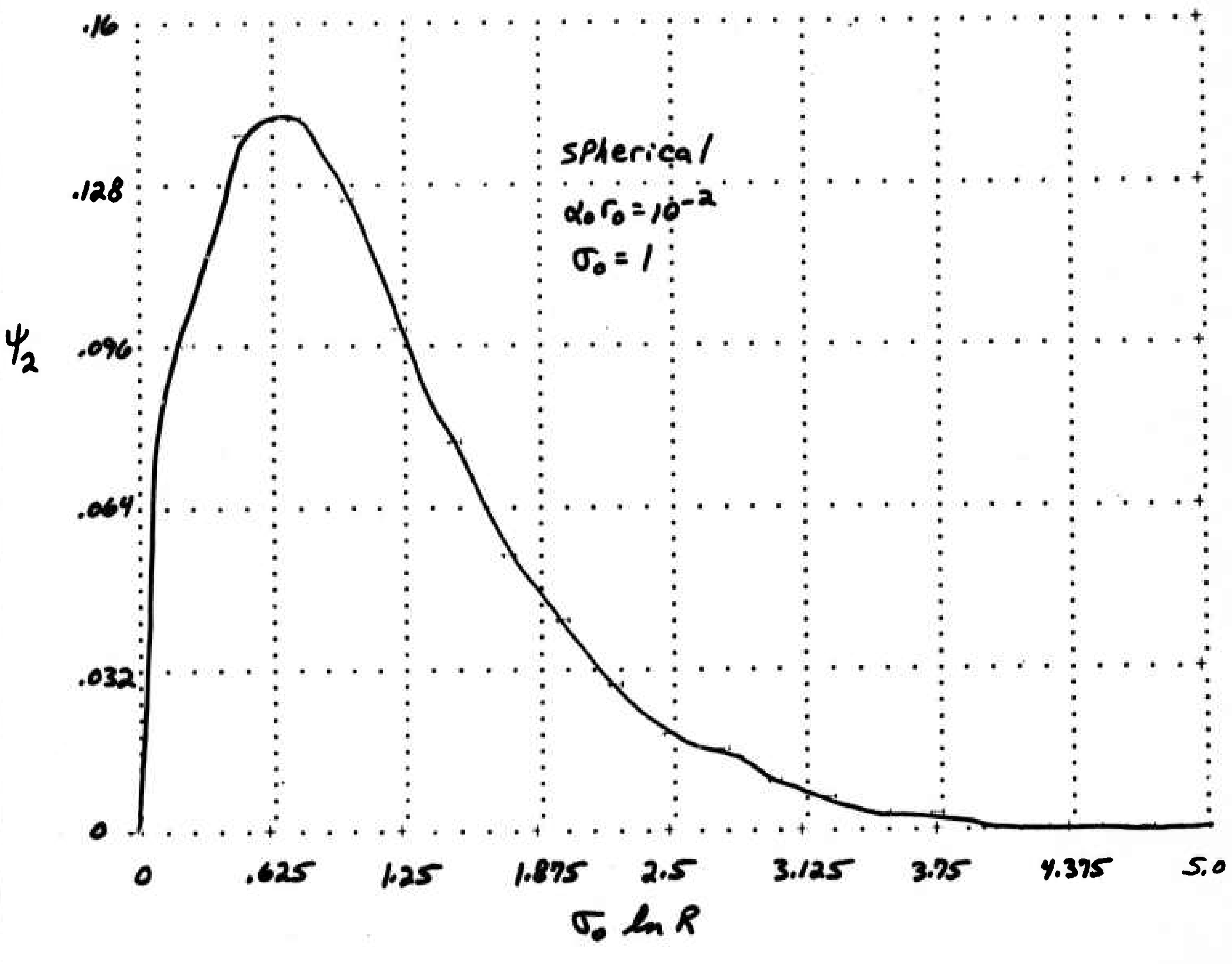
$\alpha_o r_o = 10^{-2}$

$\sigma_o = 1$

ψ_1, ψ_2, EXDB



1.0
.8
.6
.4
.2
0
ψ_1
Spherical
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
$\sigma_0 \ln R$


.16
.128
.096
.064
.032
0
ψ_2
spherical
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
$\sigma_0 \ln R$

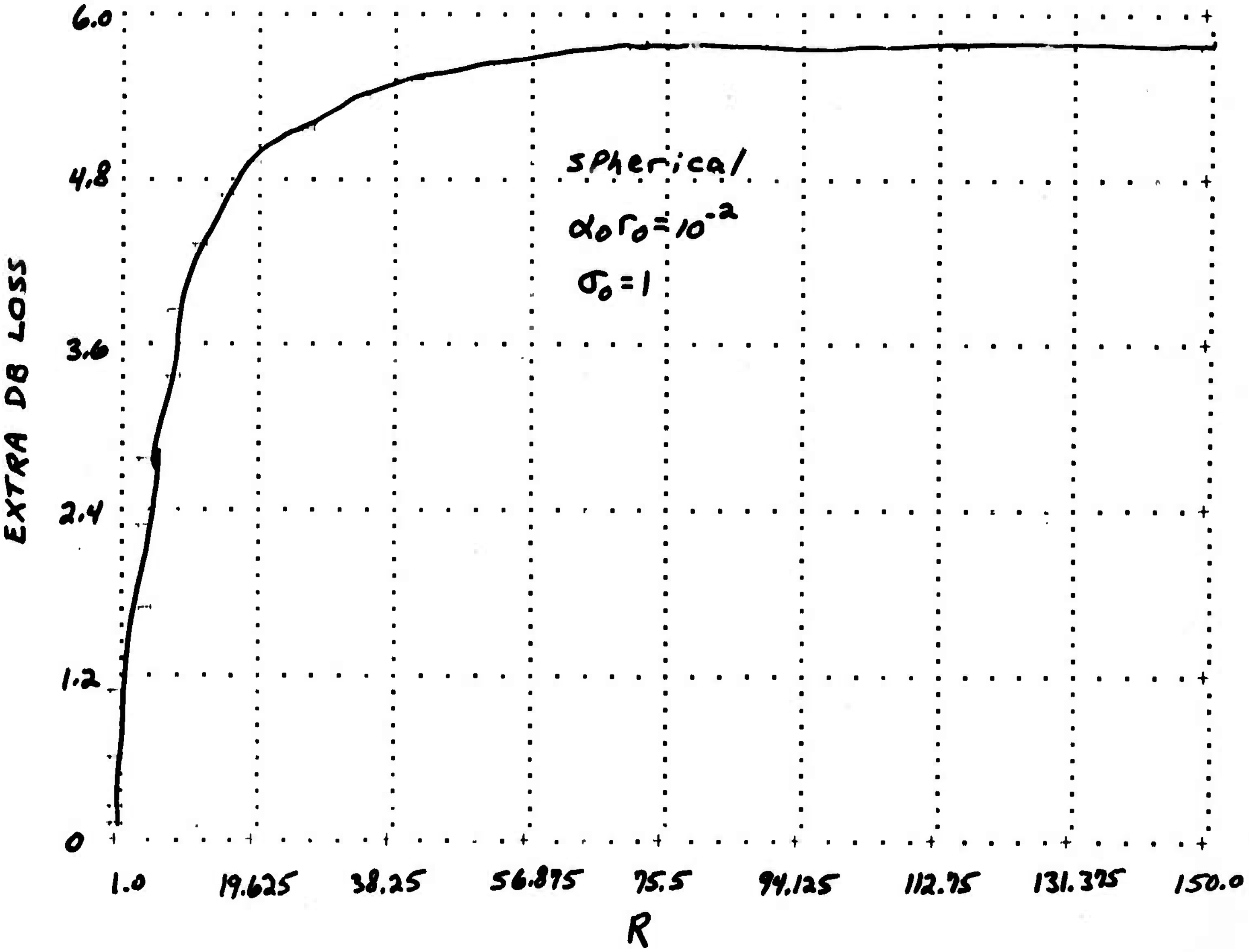

6.0
4.8
3.6
2.4
1.2
0
EXTRA DB LOSS
1.0
19.625
38.25
56.875
75.5
94.125
112.75
131.375
150.0
R
spherical
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$

Figure 5

(pages F18to F22)

Mixed case

[illegible]$_o = 10^{-4}$

[illegible]$_o = 1$

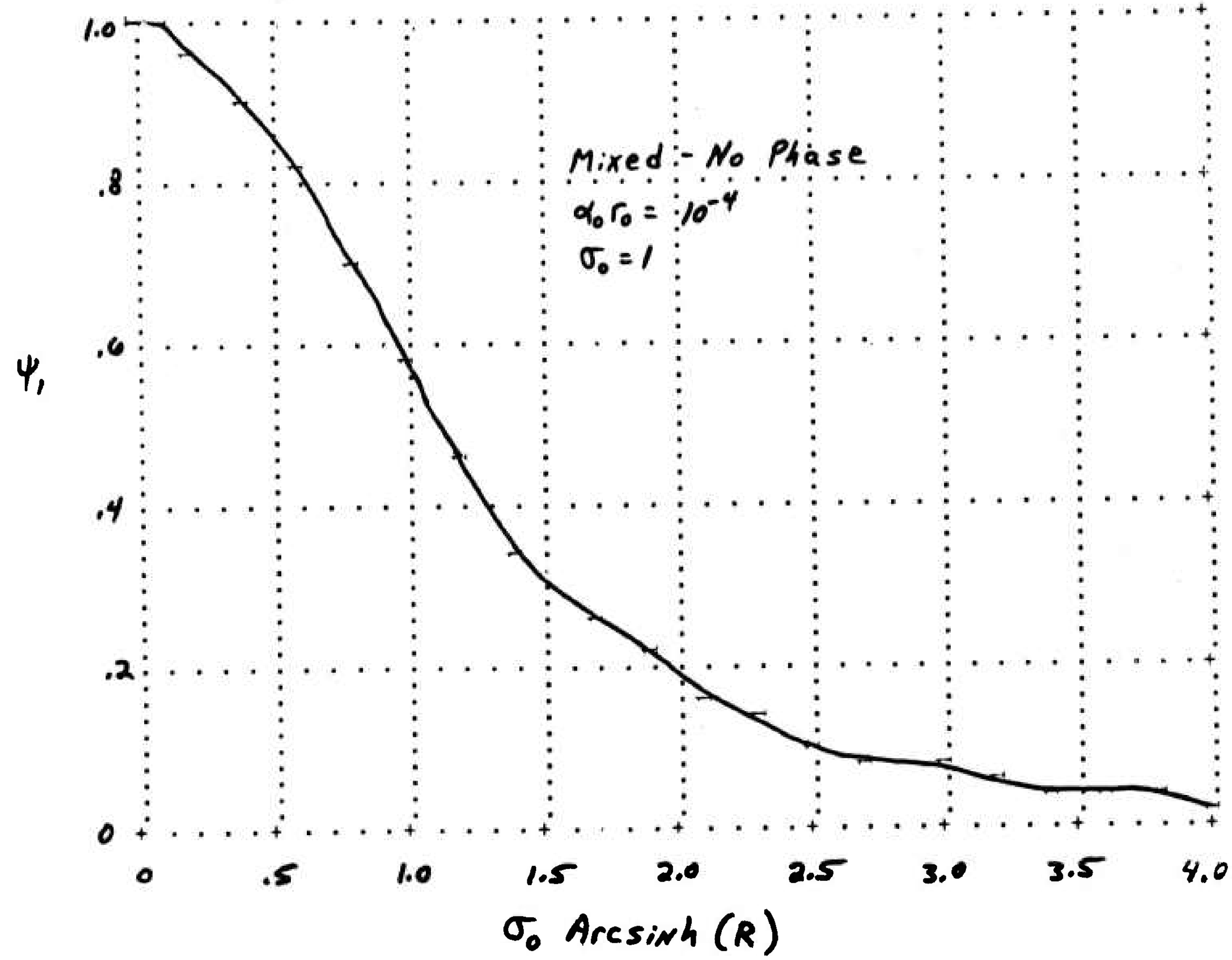
ψ_1 - mixed no phase

ψ_1 - mixed approximate E1 form

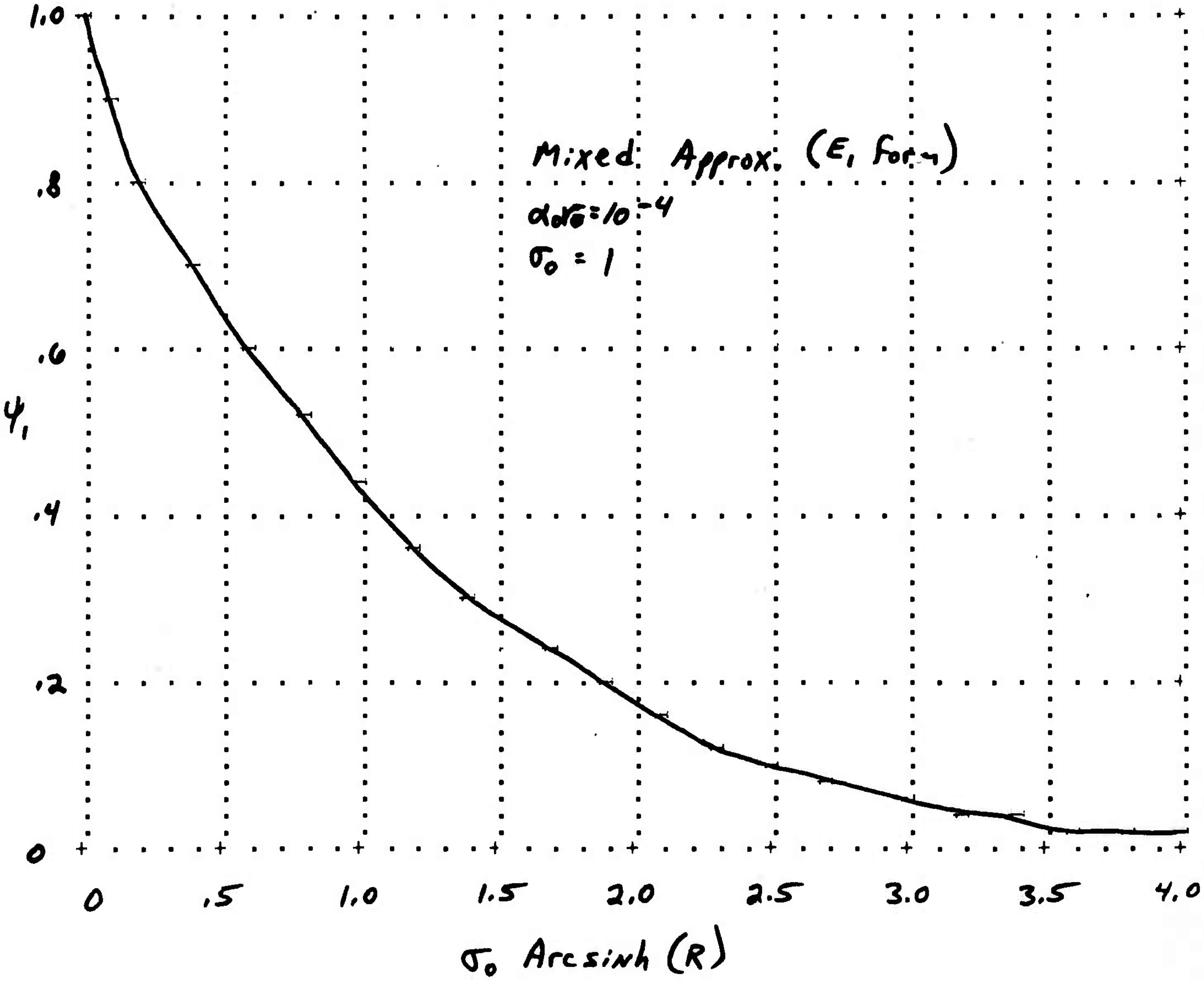
ψ_1 - mixed approximate integral form

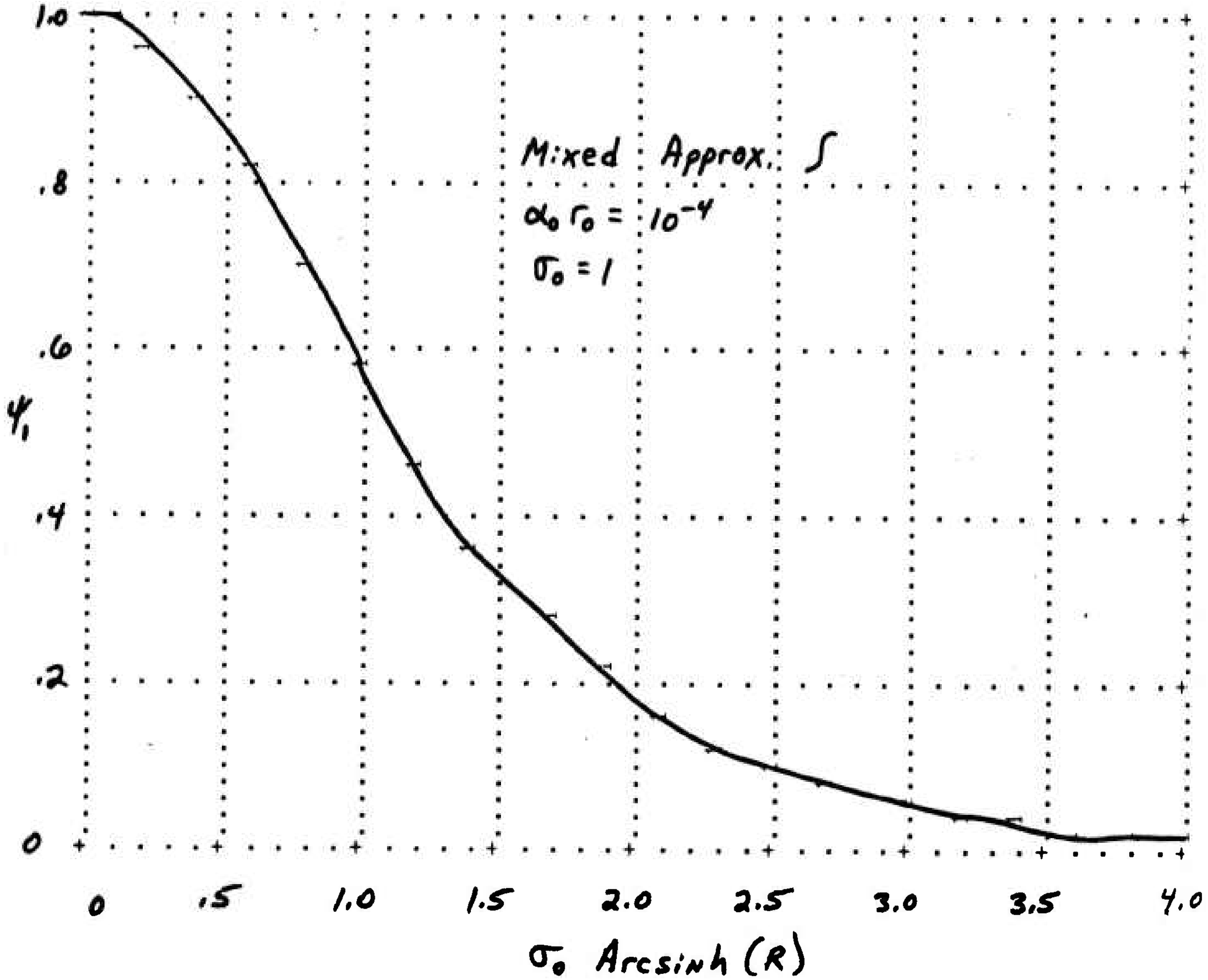
ψ_2 - mixed no phase

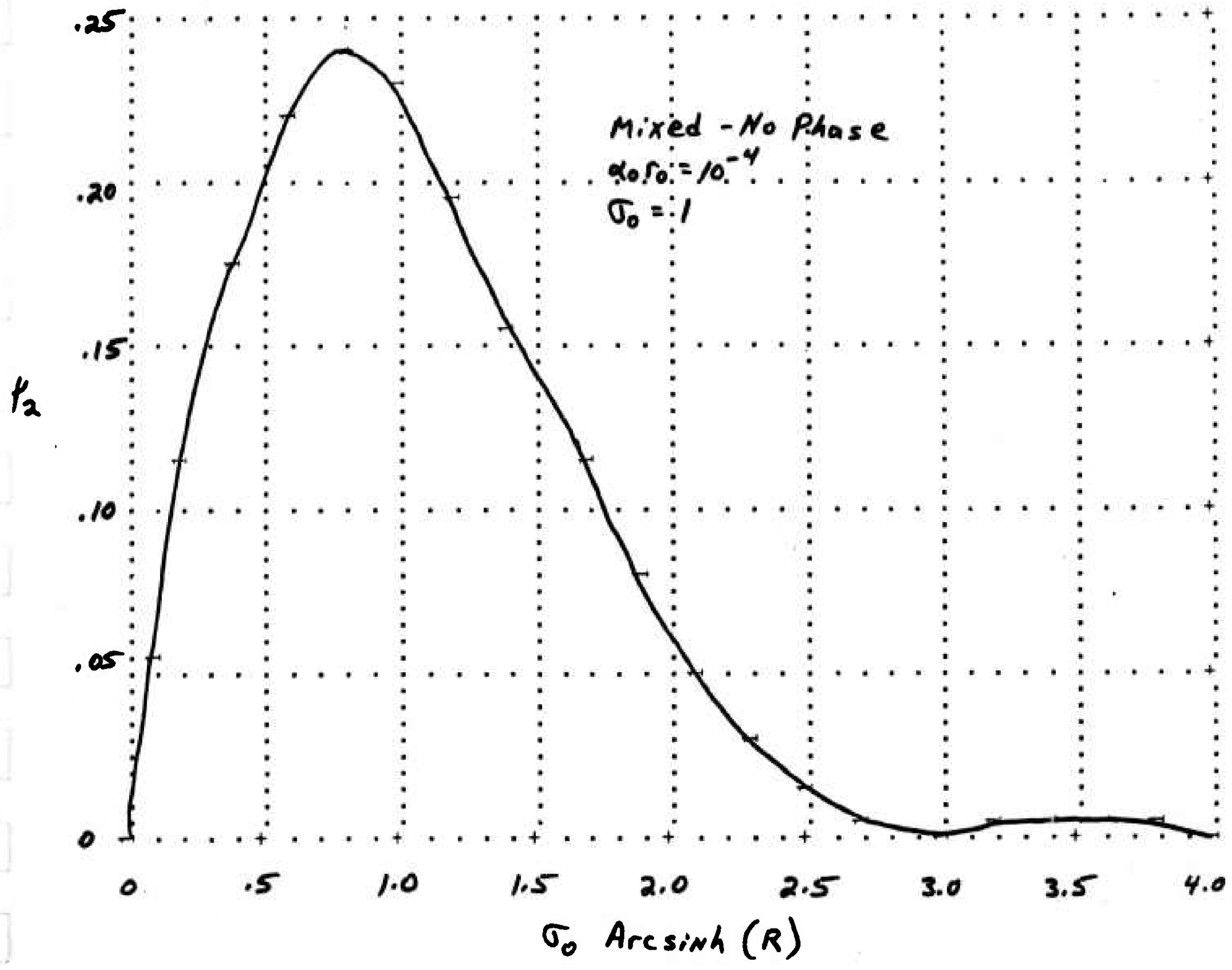
EXDB - mixed approximate integral form



1.0
.8
.6
.4
.2
0
ψ_1
Mixed - No Phase
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 1$
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
σ_0 Arcsinh (R)


Mixed Approx. (E_1 Form)
$\alpha_0\sqrt{\sigma_0}=10^{-4}$
$\sigma_0 = 1$
1.0
.8
.6
.4
.2
0
ψ_1
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
σ_0 Arcsinh (R)


Mixed Approx. $\int$
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 1$
1.0
.8
.6
.4
.2
0
ψ_1
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
σ_0 Arcsinh (R)


Mixed - No Phase
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = .1$
.25
.20
.15
.10
.05
0
ψ_2
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
σ_0 Arcsinh (R)

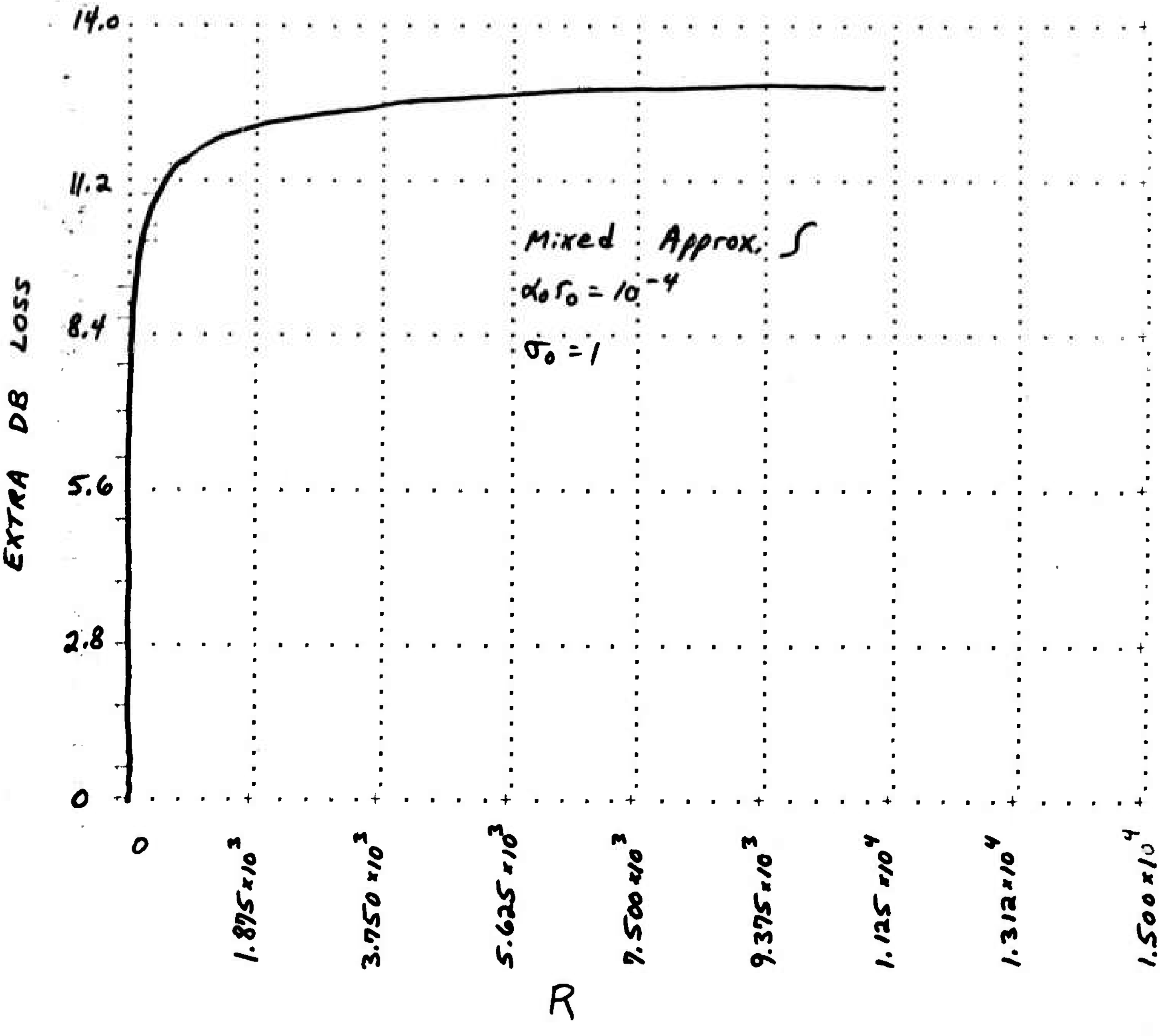

14.0
11.2
8.4
5.6
2.8
0
EXTRA DB LOSS
Mixed Approx. ʃ
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 1$
0
1.875×10^3
3.750×10^3
5.625×10^3
7.500×10^3
9.375×10^3
1.125×10^4
1.312×10^4
1.500×10^4
R

Figure 6

(pages F24to F29)

Mixed case

$\alpha_o r_o = 10^{-2}$

$\sigma_o = 1$

ψ_1 - mixed no phase

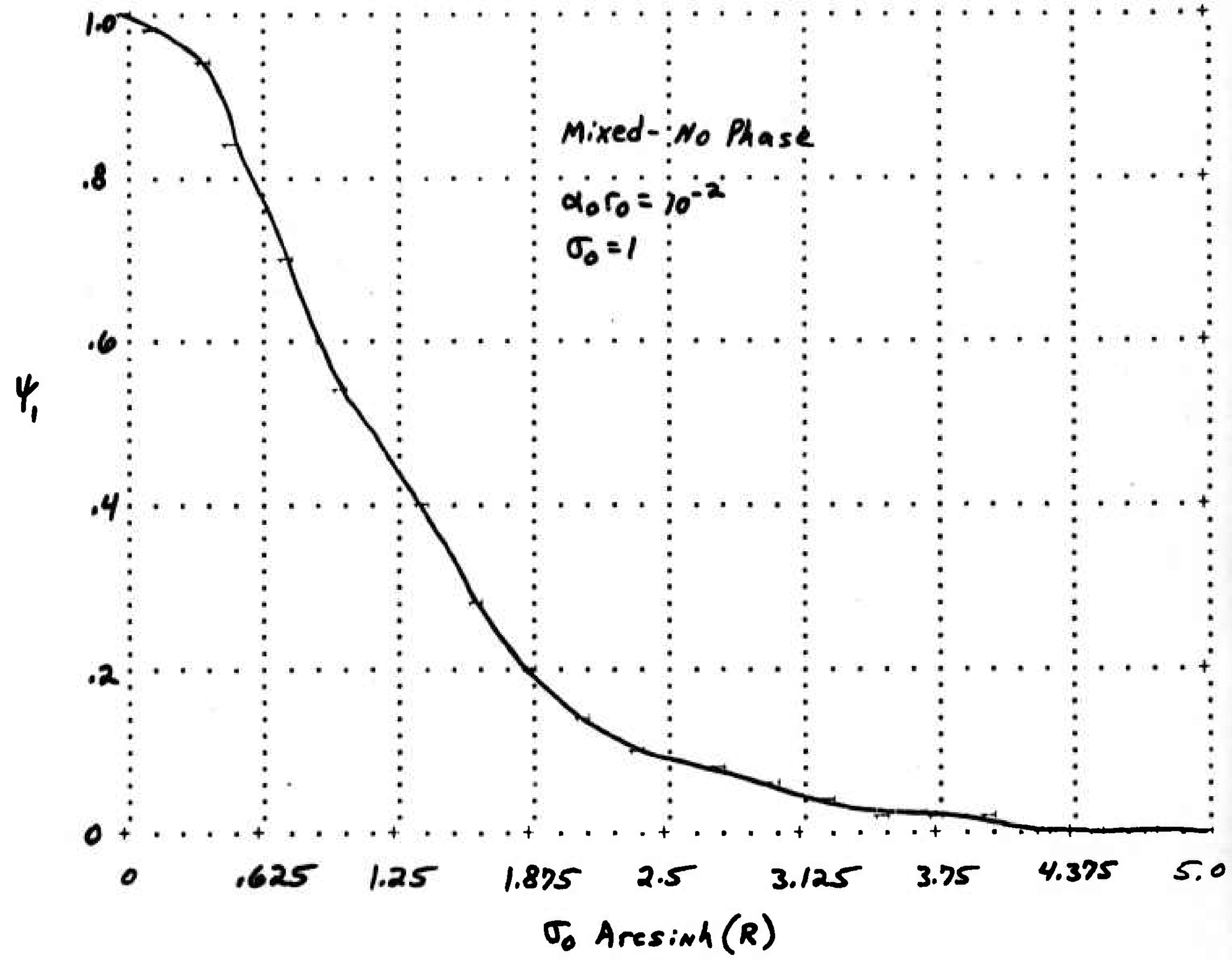
ψ_1 - mixed approximate E1 form

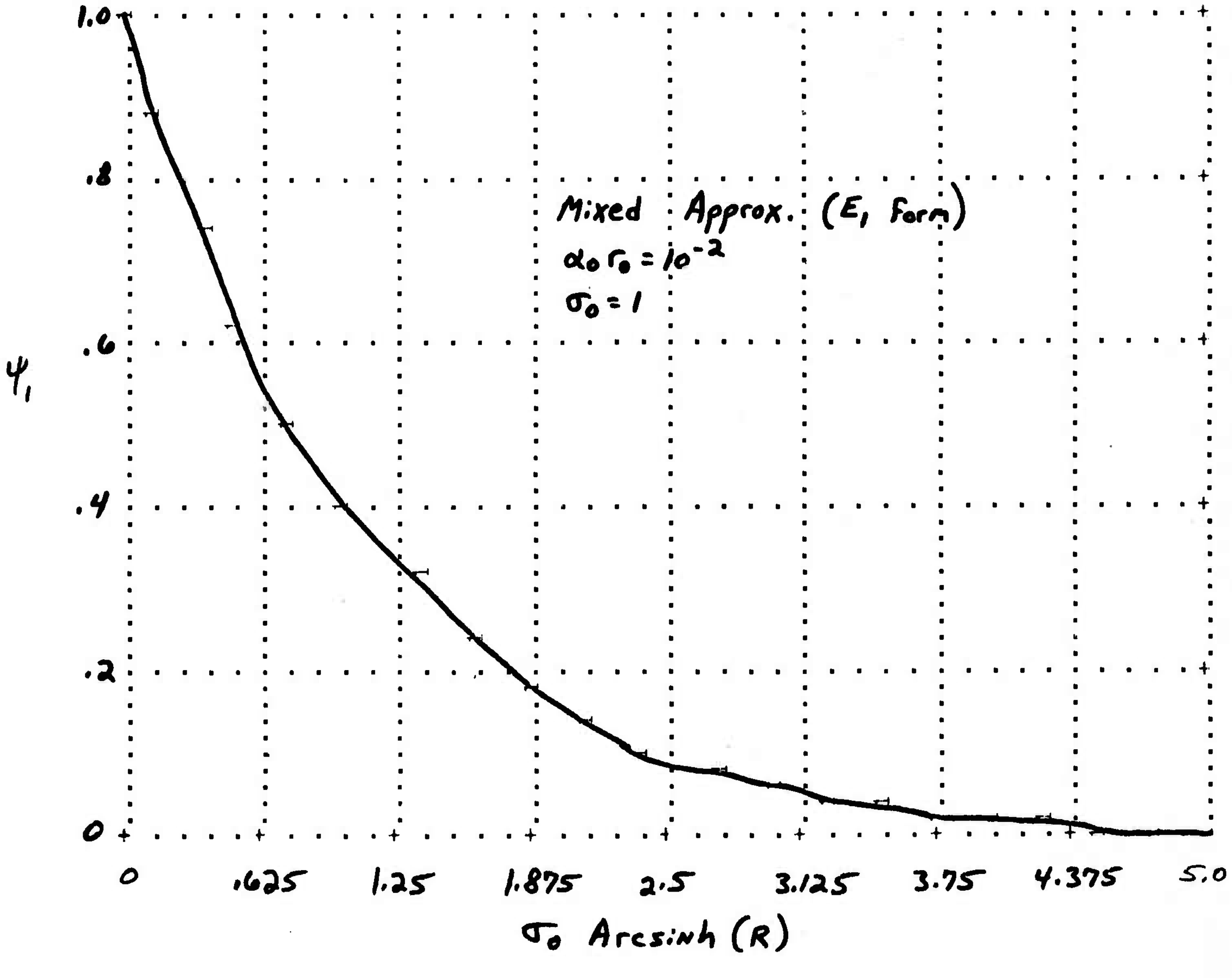
ψ_1 - mixed approximate integral form

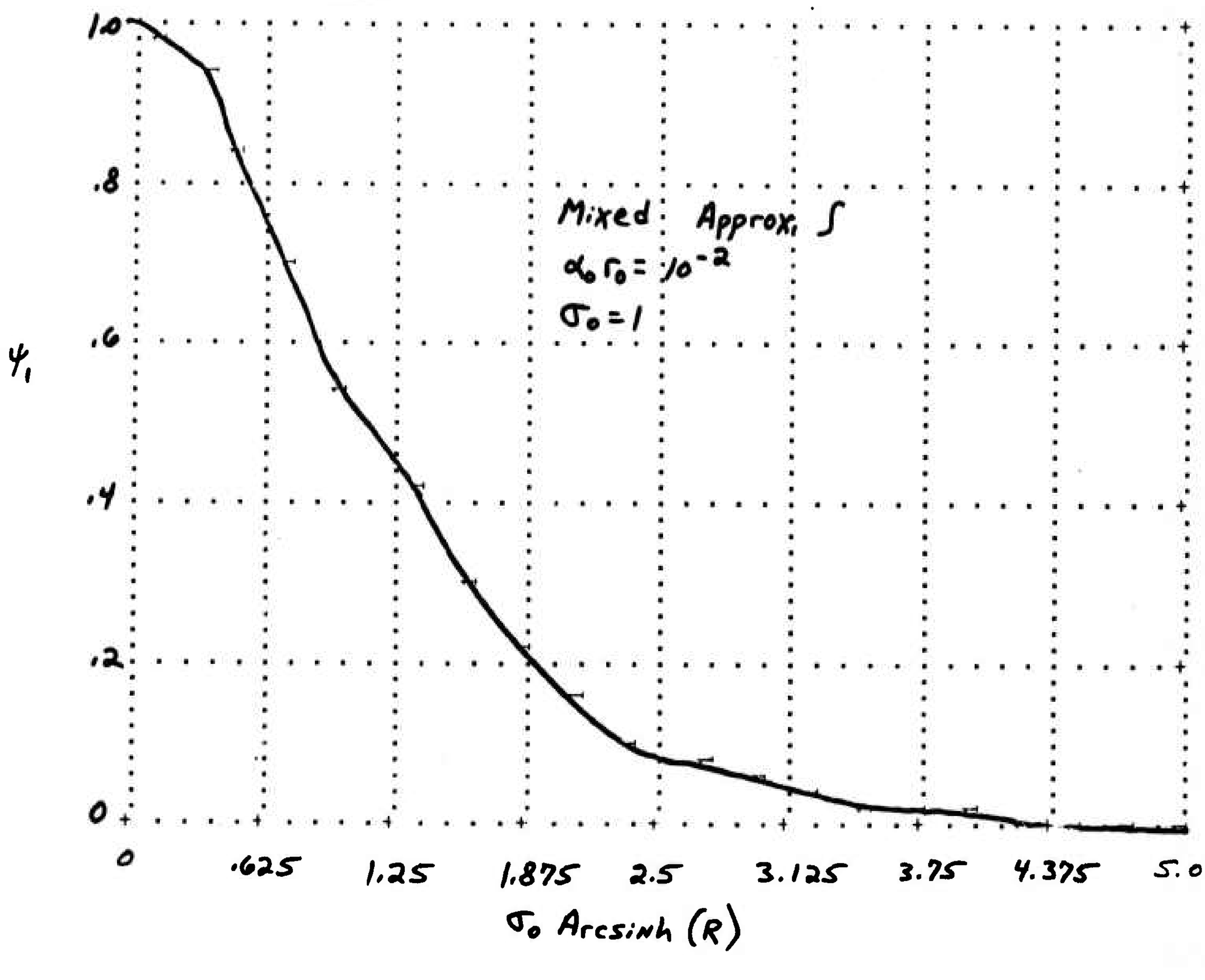
ψ_2 - mixed no phase

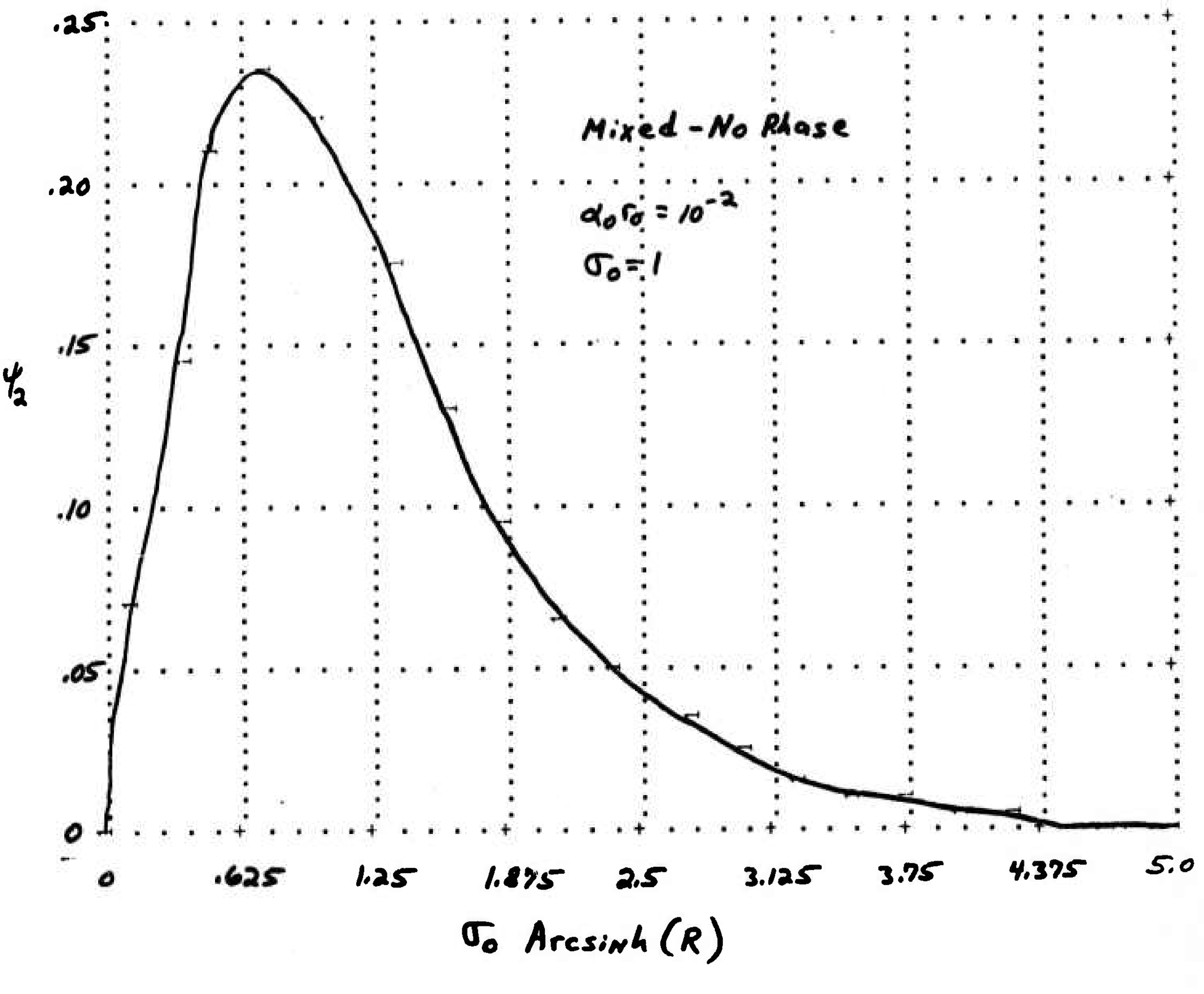
EXDB - mixed no phase

EXDB - mixed approximate integral form

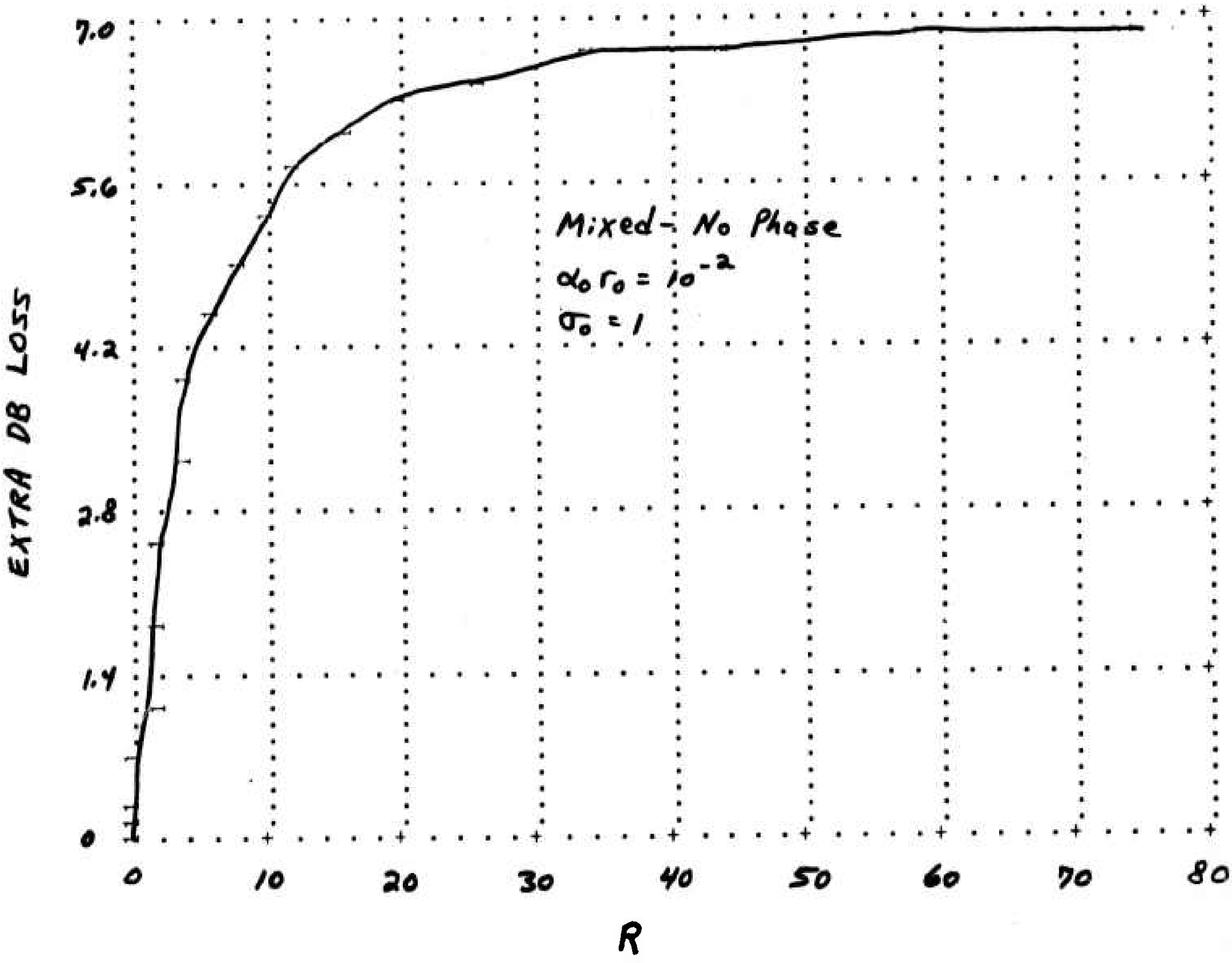

1.0
.8
.6
.4
.2
0
ψ_1
Mixed- No Phase
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
σ_0 Arcsinh(R)


1.0
.8
.6
.4
.2
0
ψ_1
Mixed Approx. (E_1 Form)
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
σ_0 Arcsinh (R)


1.0
.8
.6
.4
.2
0
ψ_1
Mixed Approx. ∫
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
σ_0 Arcsinh (R)



.25
.20
.15
.10
.05
0
ψ_2
Mixed - No Phase
$\alpha_0 \sigma_0 = 10^{-2}$
$\sigma_0 = 1$
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
σ_0 Arcsinh (R)


7.0
5.6
4.2
2.8
1.4
0
EXTRA DB LOSS
0
10
20
30
40
50
60
70
80
R
Mixed - No Phase
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$

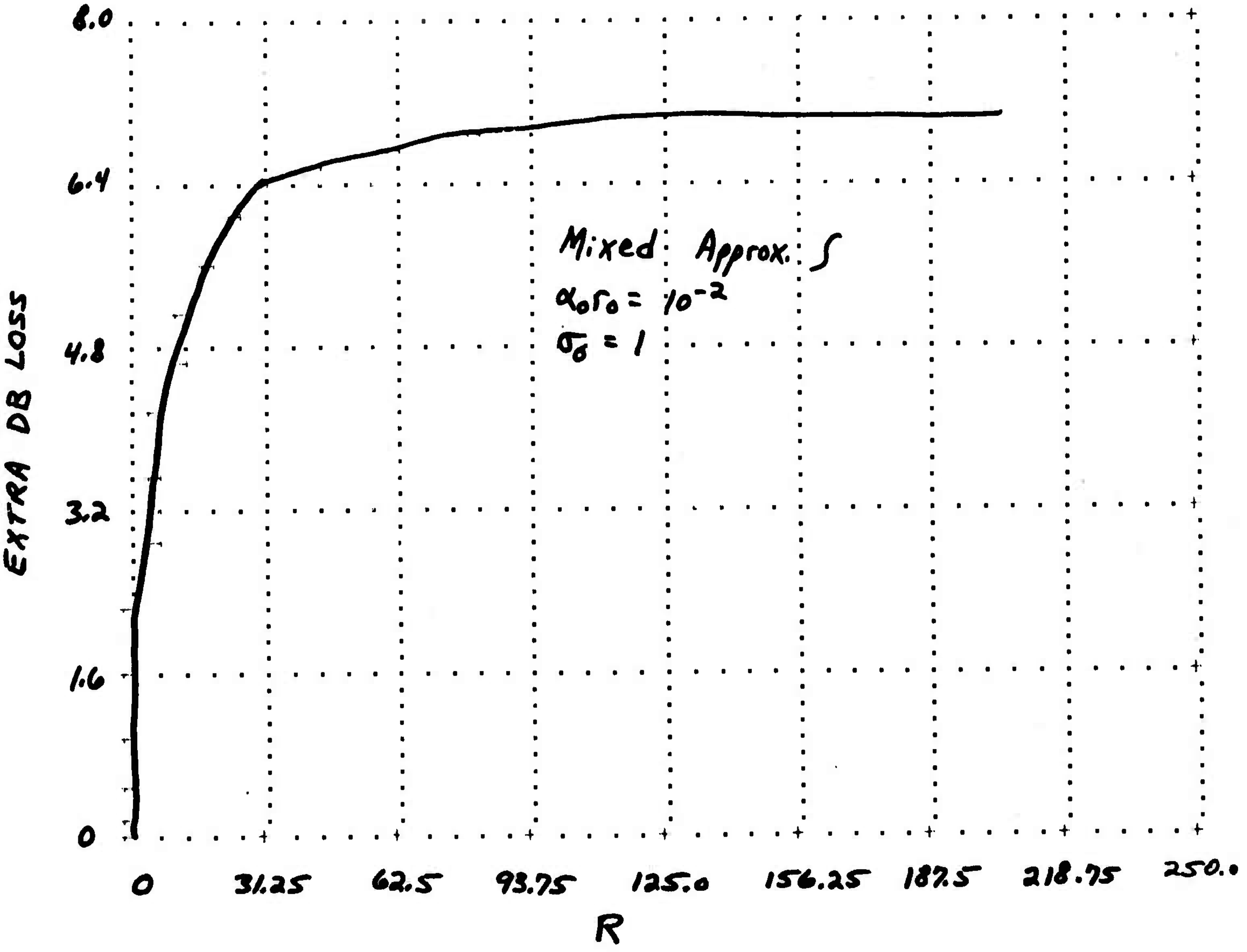

8.0
6.4
4.8
3.2
1.6
0
EXTRA DB LOSS
0
31.25
62.5
93.75
125.0
156.25
187.5
218.75
250.0
R
Mixed Approx. ∫
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$

Figure 7

(pages F31to F38)

Mixed case

$\alpha_o r_o = 7.35 \times 10^{-3}$

σ_o = 0.1707, 0.3035, 0.5398,
0.9600, 1.7071

ψ_1 - mixed no phase

ψ_1 - mixed phase

ψ_1 - mixed approximate integral form

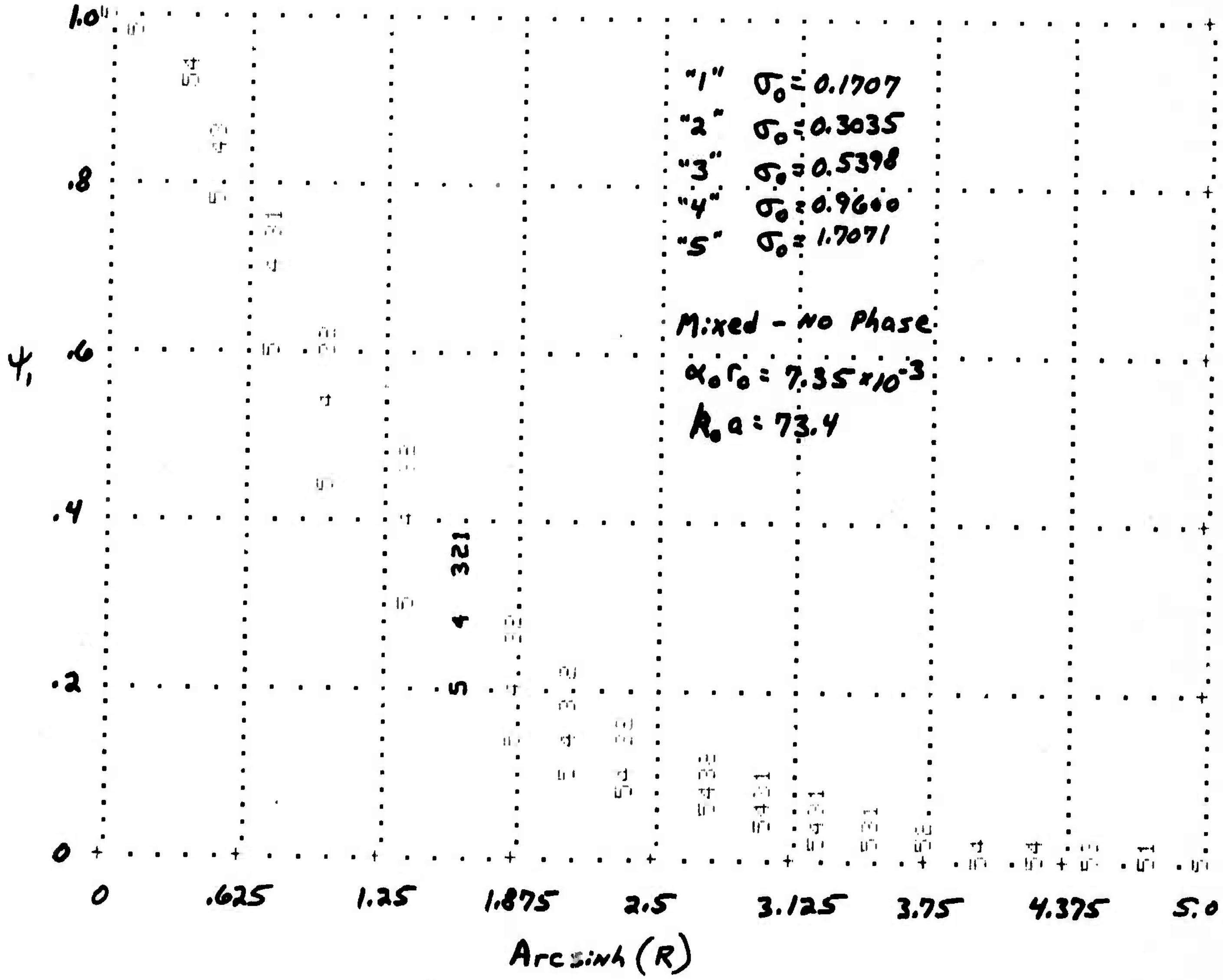
ψ_2 - mixed no phase

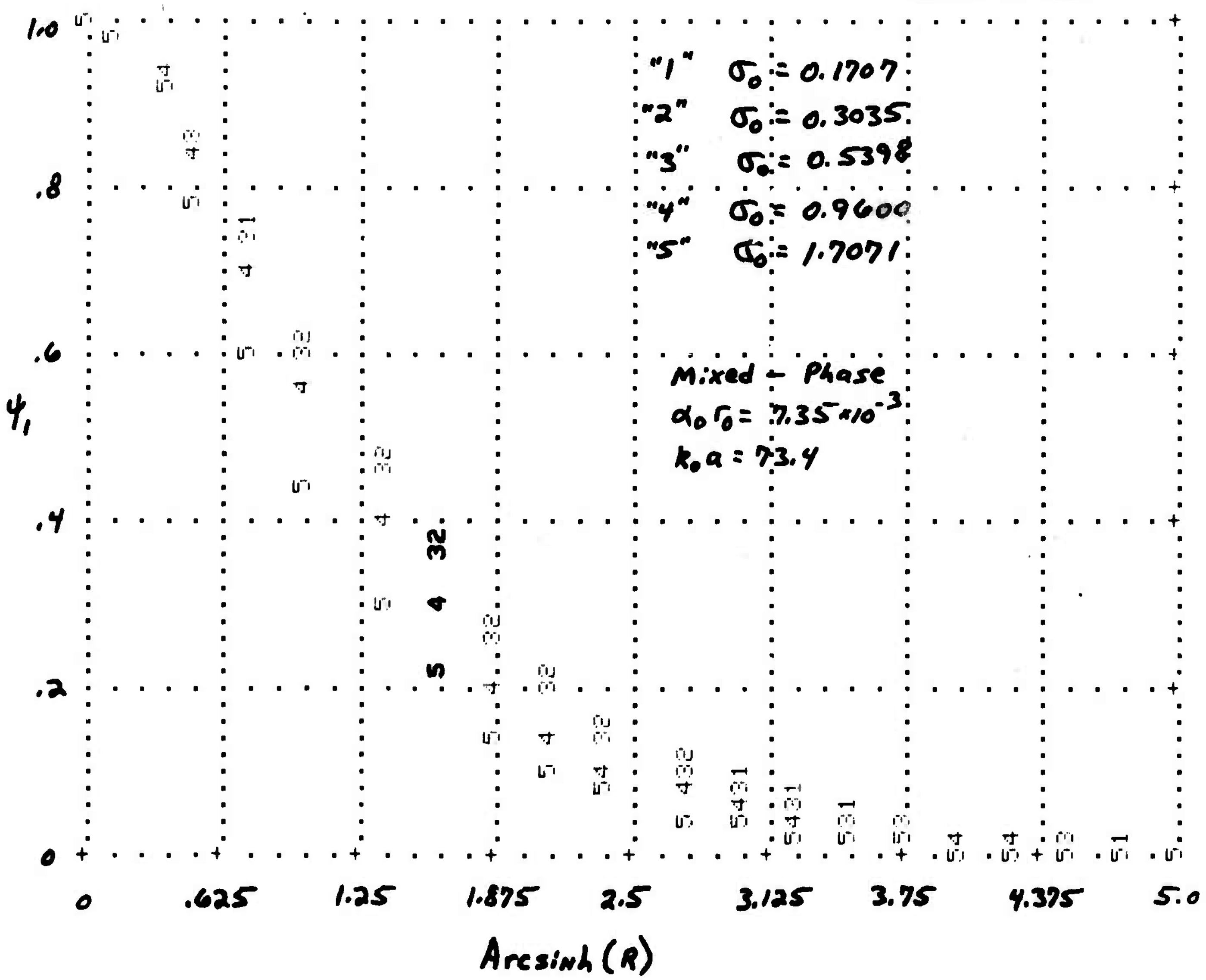
ψ_2 - mixed phase

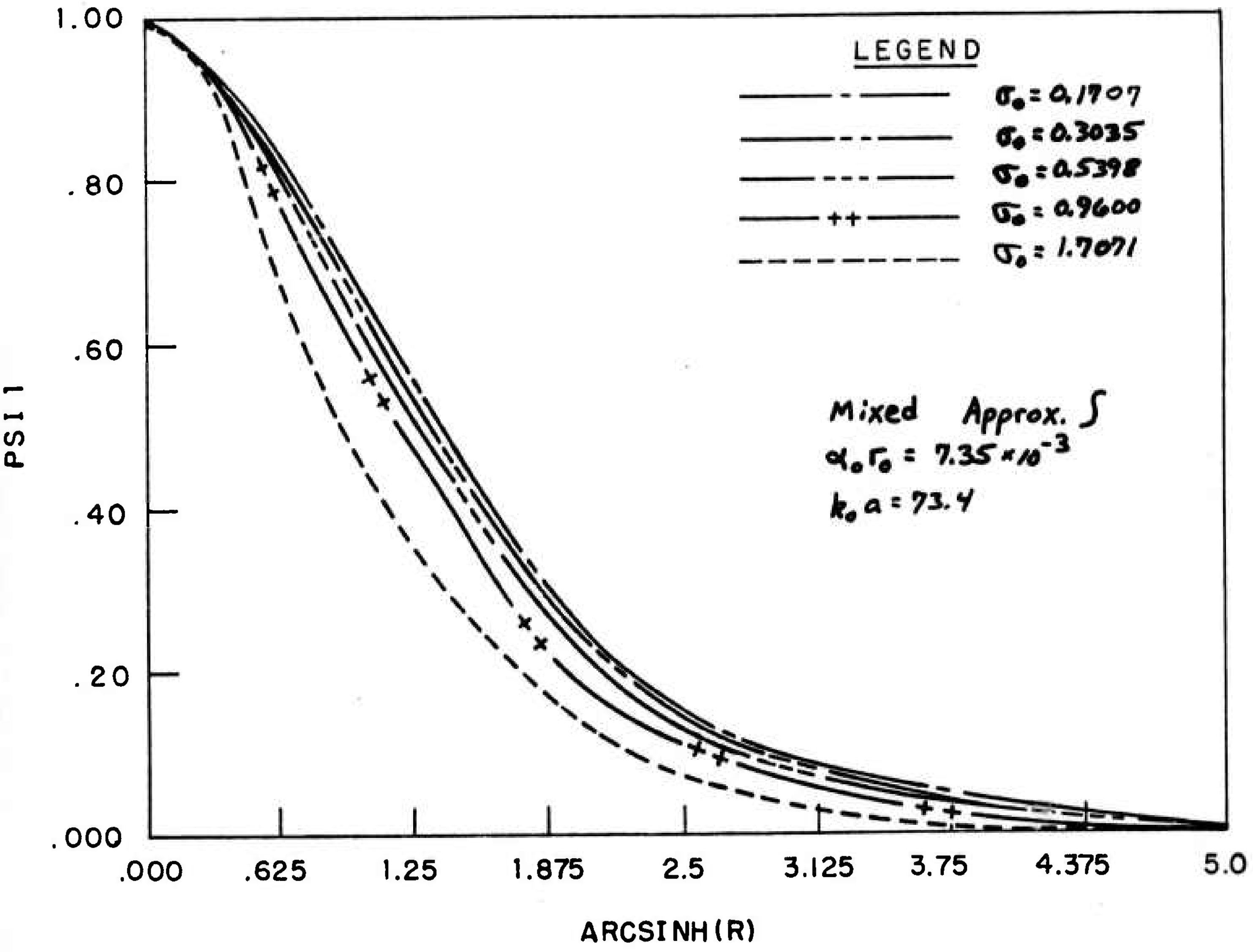
EXDB - mixed no phase

EXDB - mixed phase

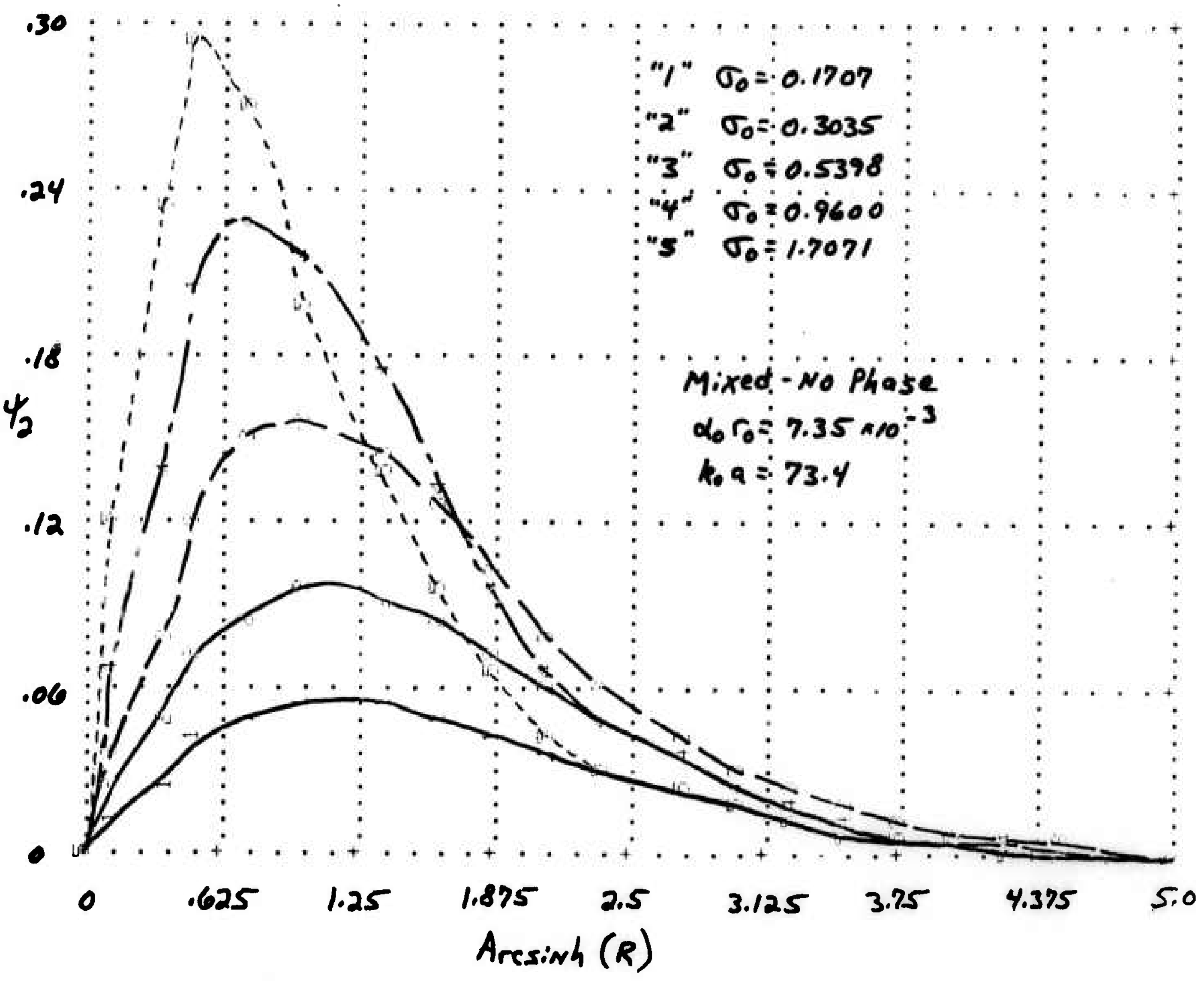
EXDB - mixed approximate integral form

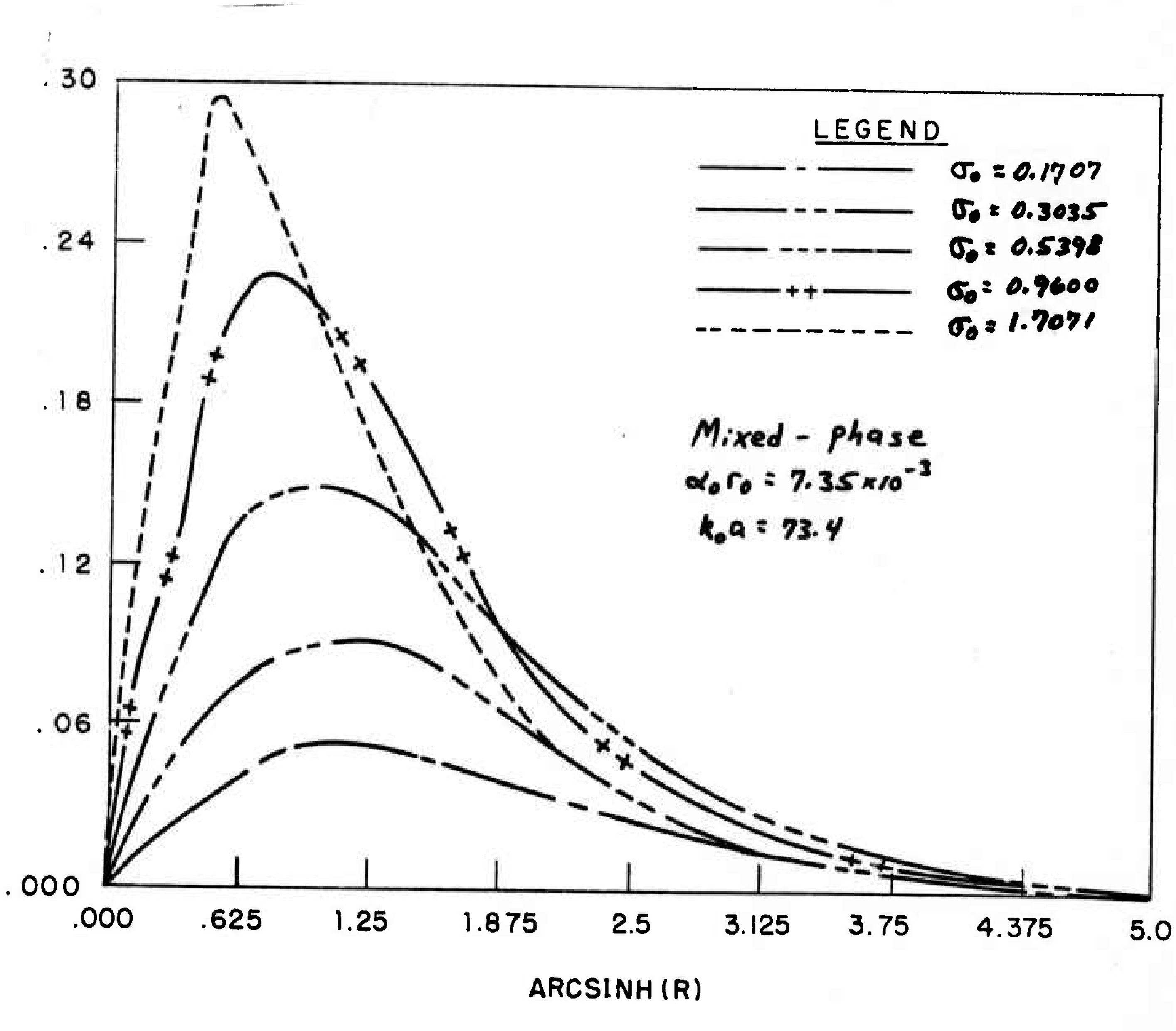

1.0
.8
.6
.4
.2
0
ψ_1
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
Arcsinh (R)
"1" σ_0 = 0.1707
"2" σ_0 = 0.3035
"3" σ_0 = 0.5398
"4" σ_0 = 0.9600
"5" σ_0 = 1.7071
Mixed - No Phase
$\alpha_0 r_0$ = 7.35 × 10^{-3}
$k_0 a$ = 73.4
321
4
5


1.0
.8
.6
.4
.2
0
ψ_1
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
Arcsinh (R)
"1" σ_0 = 0.1707
"2" σ_0 = 0.3035
"3" σ_0 = 0.5398
"4" σ_0 = 0.9600
"5" σ_0 = 1.7071
Mixed - Phase
$\alpha_0 r_0$ = 7.35×10^{-3}
$k_0 a$ = 73.4

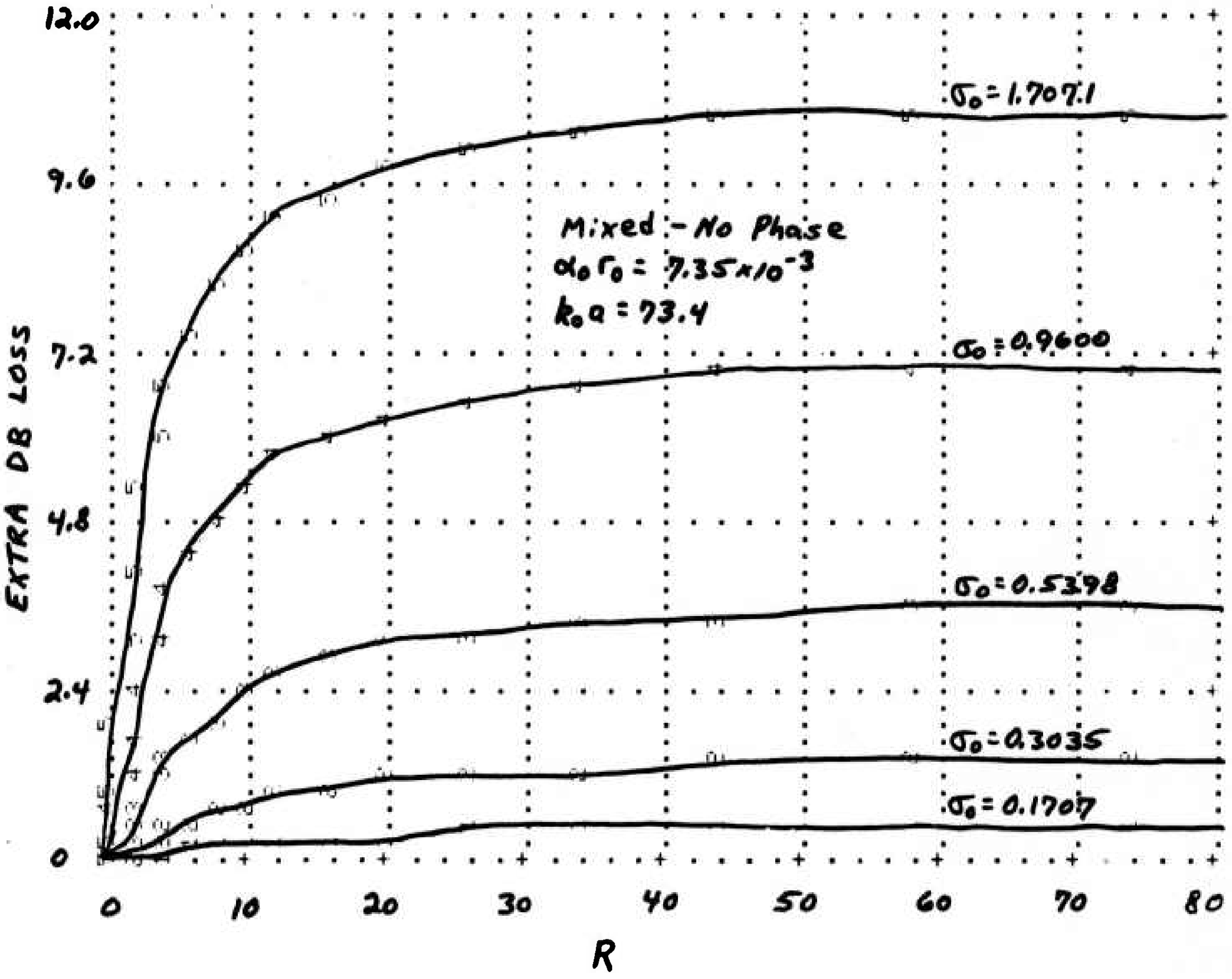


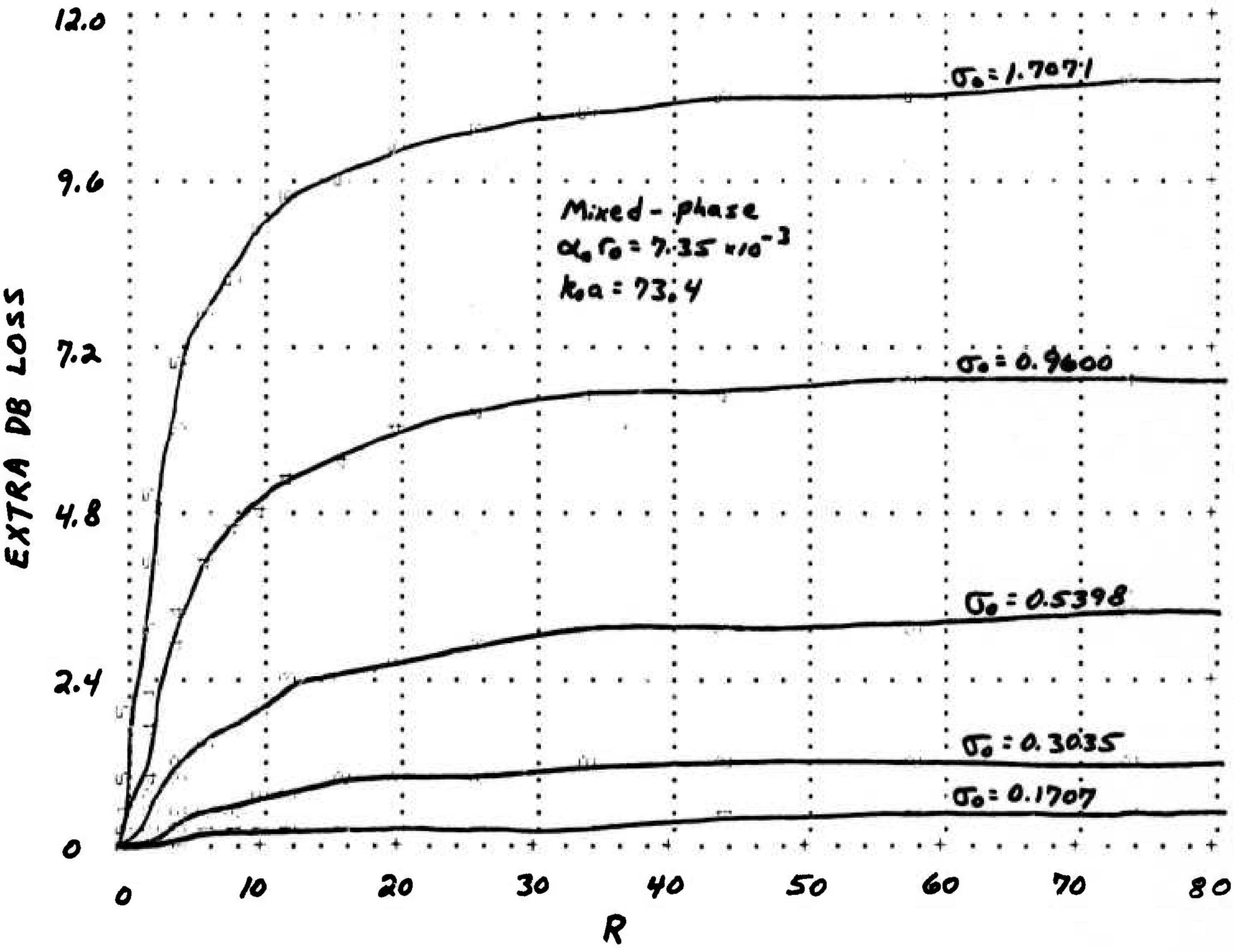
1.00
.80
.60
.40
.20
.000
PSI1
.000
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
ARCSINH(R)
LEGEND
$\sigma_0 = 0.1707$
$\sigma_0 = 0.3035$
$\sigma_0 = 0.5398$
$\sigma_0 = 0.9600$
$\sigma_0 = 1.7071$
Mixed Approx. ∫
$\alpha_0 r_0 = 7.35 \times 10^{-3}$
$k_0 a = 73.4$


.30
.24
.18
.12
.06
0
ψ_2
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
Arcsinh (R)
"1" σ_0 = 0.1707
"2" σ_0 = 0.3035
"3" σ_0 = 0.5398
"4" σ_0 = 0.9600
"5" σ_0 = 1.7071
Mixed - No Phase
$\alpha_0 r_0 = 7.35 \times 10^{-3}$
$k_0 a = 73.4$



.30
.24
.18
.12
.06
.000
PSI 2
.000
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
ARCSINH (R)
LEGEND
$\sigma_0 = 0.1707$
$\sigma_0 = 0.3035$
$\sigma_0 = 0.5398$
$\sigma_0 = 0.9600$
$\sigma_0 = 1.7071$
Mixed - phase
$\alpha_0 r_0 = 7.35 \times 10^{-3}$
$k_0 a = 73.4$


12.0
9.6
7.2
4.8
2.4
0
EXTRA DB LOSS
0
10
20
30
40
50
60
70
80
R
σ_0 = 1.7071
σ_0 = 0.9600
σ_0 = 0.5398
σ_0 = 0.3035
σ_0 = 0.1707
Mixed - No Phase
$\alpha_0 r_0 = 7.35 \times 10^{-3}$
$k_0 a = 73.4$


12.0
9.6
7.2
4.8
2.4
0
EXTRA DB LOSS
0
10
20
30
40
50
60
70
80
R
$\sigma_o = 1.7071$
$\sigma_o = 0.9600$
$\sigma_o = 0.5398$
$\sigma_o = 0.3035$
$\sigma_o = 0.1707$
Mixed - phase
$\alpha_o r_o = 7.35 \times 10^{-3}$
$k_o a = 73.4$

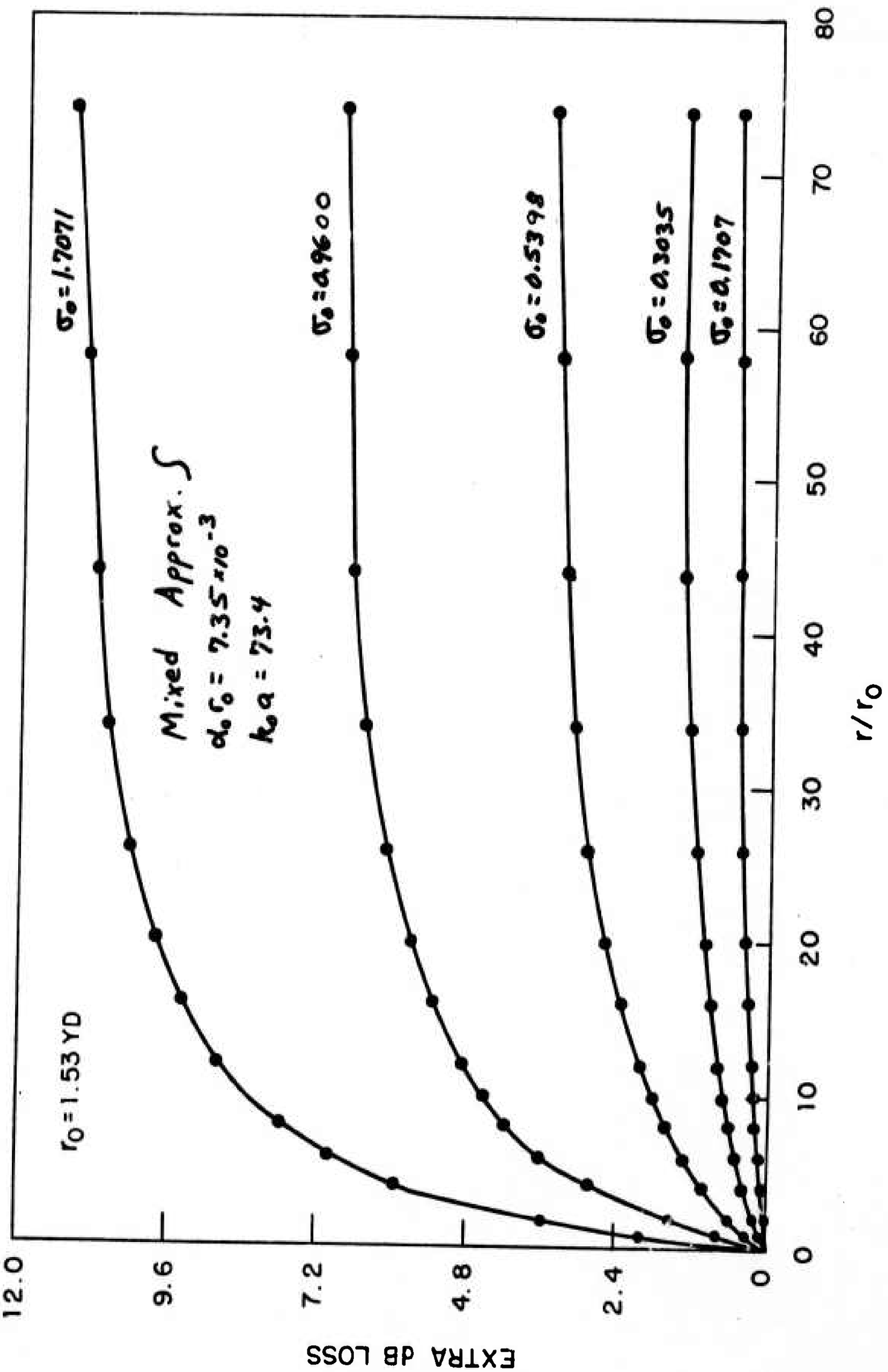

EXTRA dB LOSS
12.0
9.6
7.2
4.8
2.4
0
0
10
20
30
40
50
60
70
80
r/r₀
r₀=1.53 YD
Mixed Approx. ∫
α₀r₀ = 7.35×10⁻³
k₀a = 73.4
σ₀=1.7071
σ₀=0.9600
σ₀=0.5398
σ₀=0.3035
σ₀=0.1707

Figure 8

(page F40)

Spherical and mixed cases

$\sigma_o/\alpha_o r_o = 727$

$\sigma_o = 0.1538$

ψ_2/ψ_1

ψ_3/ψ_1

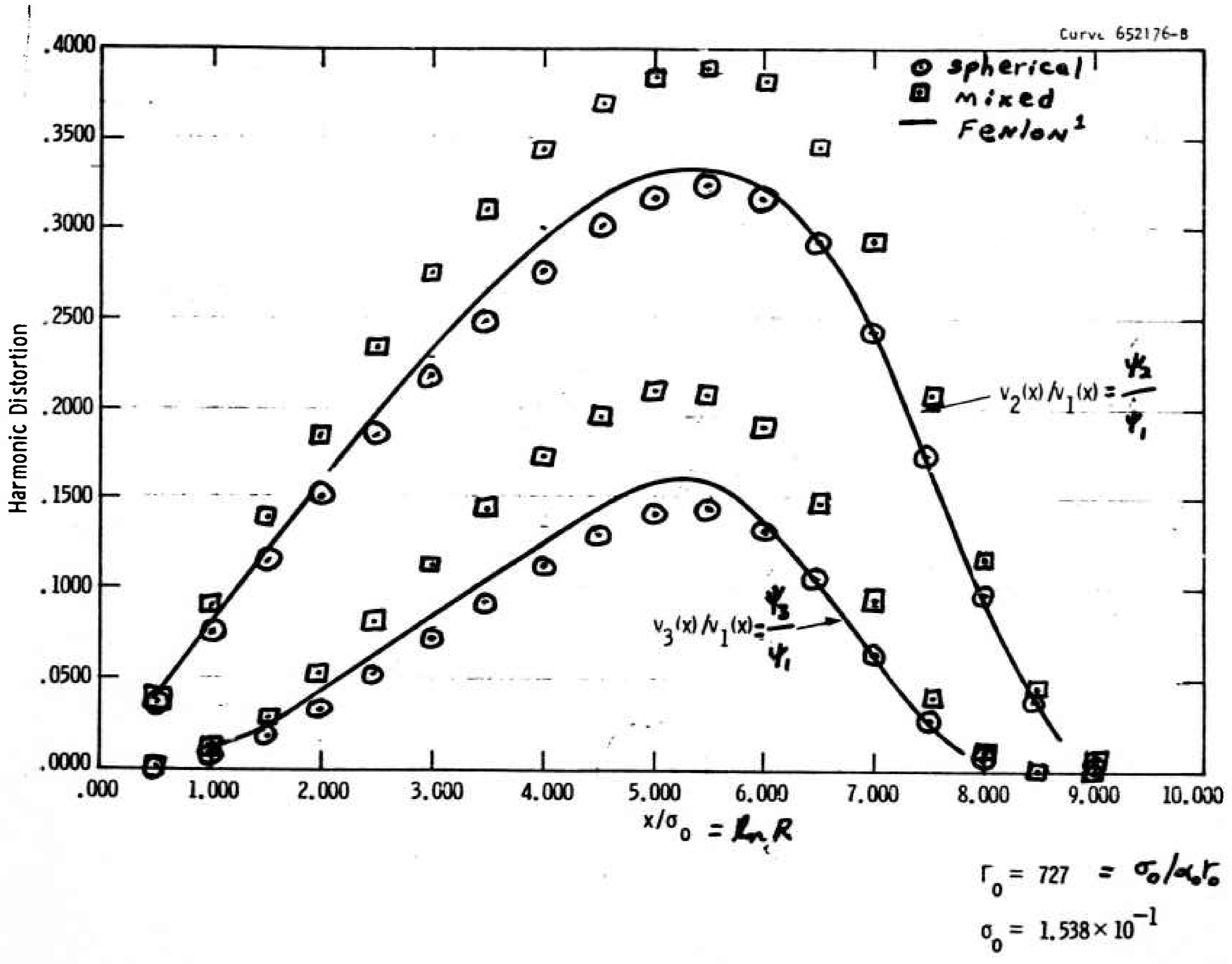

Curve 652176-B
spherical
mixed
Fenlon 1
Harmonic Distortion
.0000
.0500
.1000
.1500
.2000
.2500
.3000
.3500
.4000
.000
1.000
2.000
3.000
4.000
5.000
6.000
7.000
8.000
9.000
10.000
x/σ_0 = $k_n R$
$v_2(x)/v_1(x) = \frac{\psi_2}{\psi_1}$
$v_3(x)/v_1(x) = \frac{\psi_3}{\psi_1}$
Γ_0 = 727 = $\sigma_0/\alpha_0 r_0$
$\sigma_0 = 1.538 \times 10^{-1}$

Figure 9

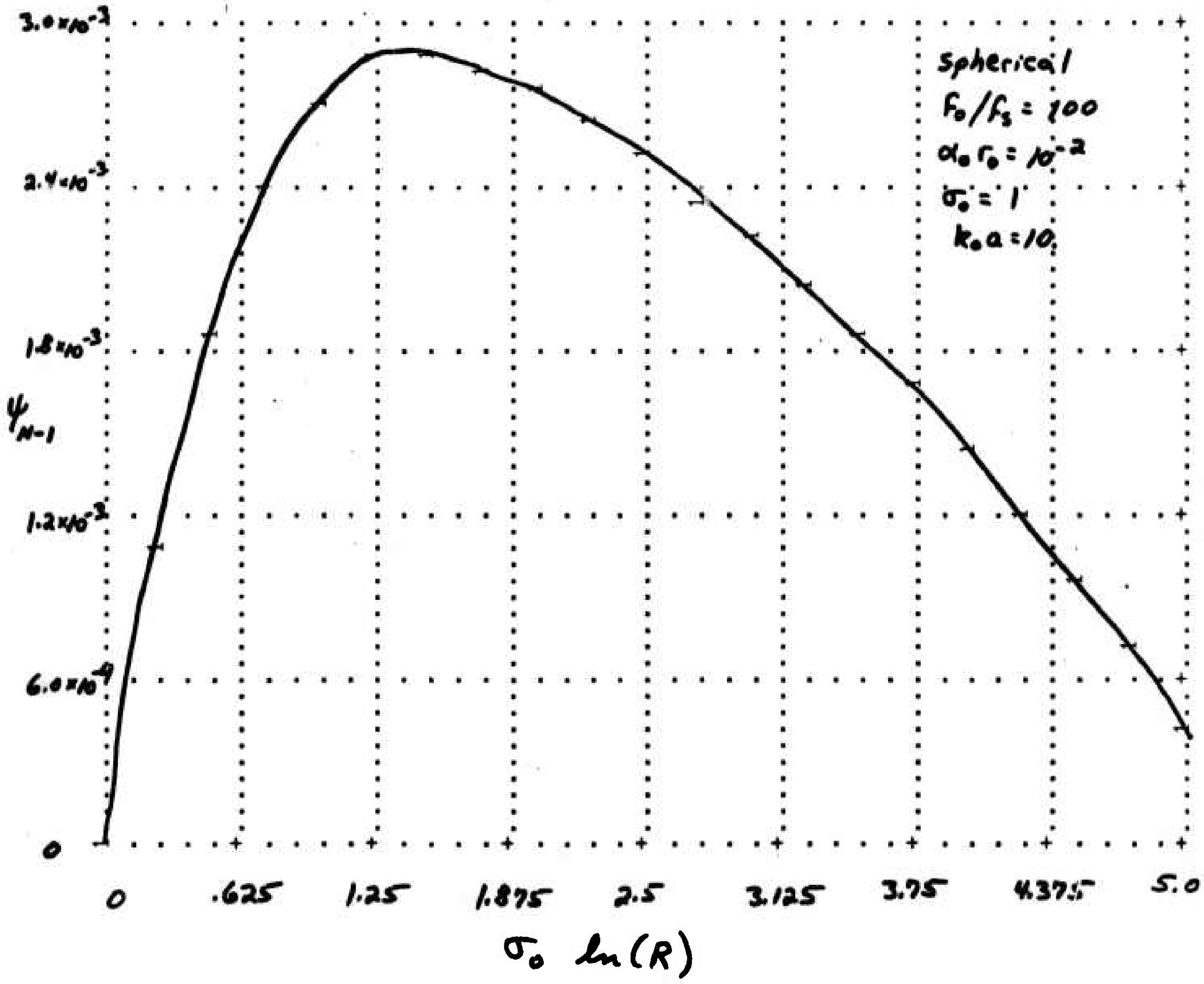
(pages F42to F43)

Spherical wave case

$\alpha_o r_o = 10^{-2}$

$\sigma_o = 1$

ψ_{NFR-1}, ψ_{NFR+1}


3.0×10⁻³
2.4×10⁻³
1.8×10⁻³
1.2×10⁻³
6.0×10⁻⁴
0
ψ_{N-1}
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
$\sigma_0 \ ln(R)$
Spherical
$f_0/f_s = 100$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
$k_0 a = 10.$

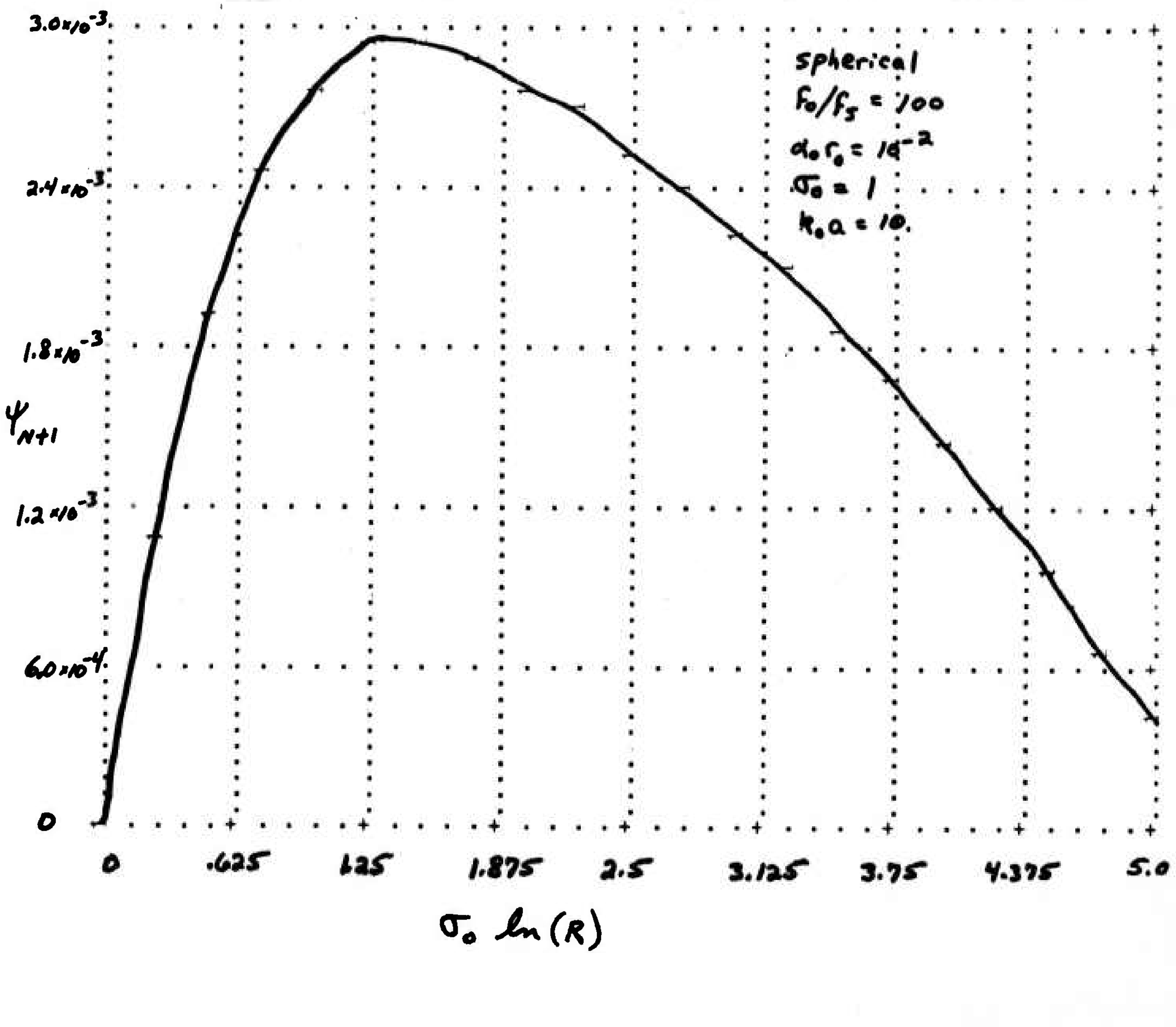

3.0×10⁻³
2.4×10⁻³
1.8×10⁻³
1.2×10⁻³
6.0×10⁻⁴
0
ψ_{N+1}
0
.625
1.25
1.875
2.5
3.125
3.75
4.375
5.0
$\sigma_0 \ln(R)$
spherical
$f_0/f_s = 100$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
$k_0 a = 10.$

Figure 10

(pagesF45 to F47)

Mixed case

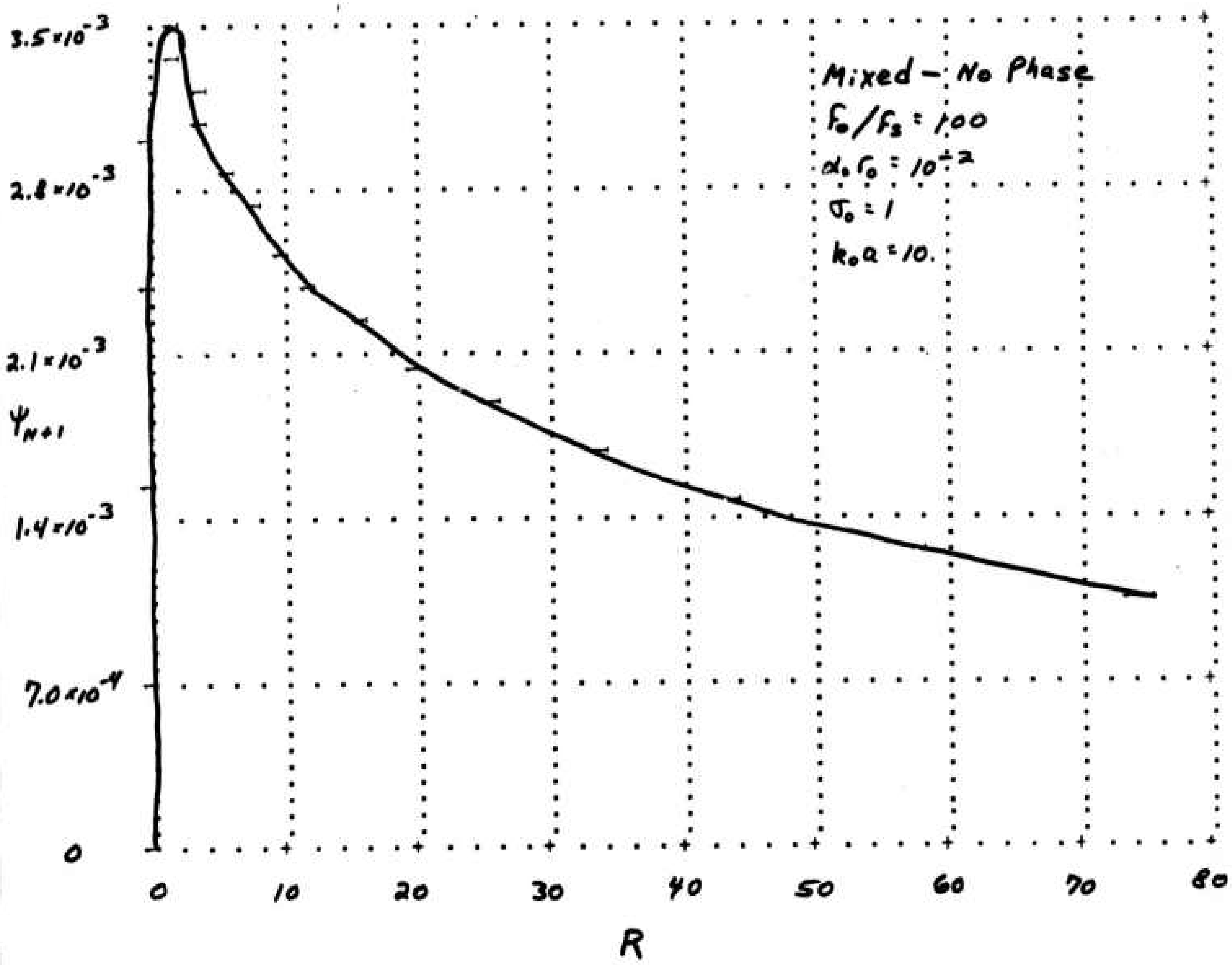
$\alpha_o r_o = 10^{-2}$

$\sigma_o = 1$

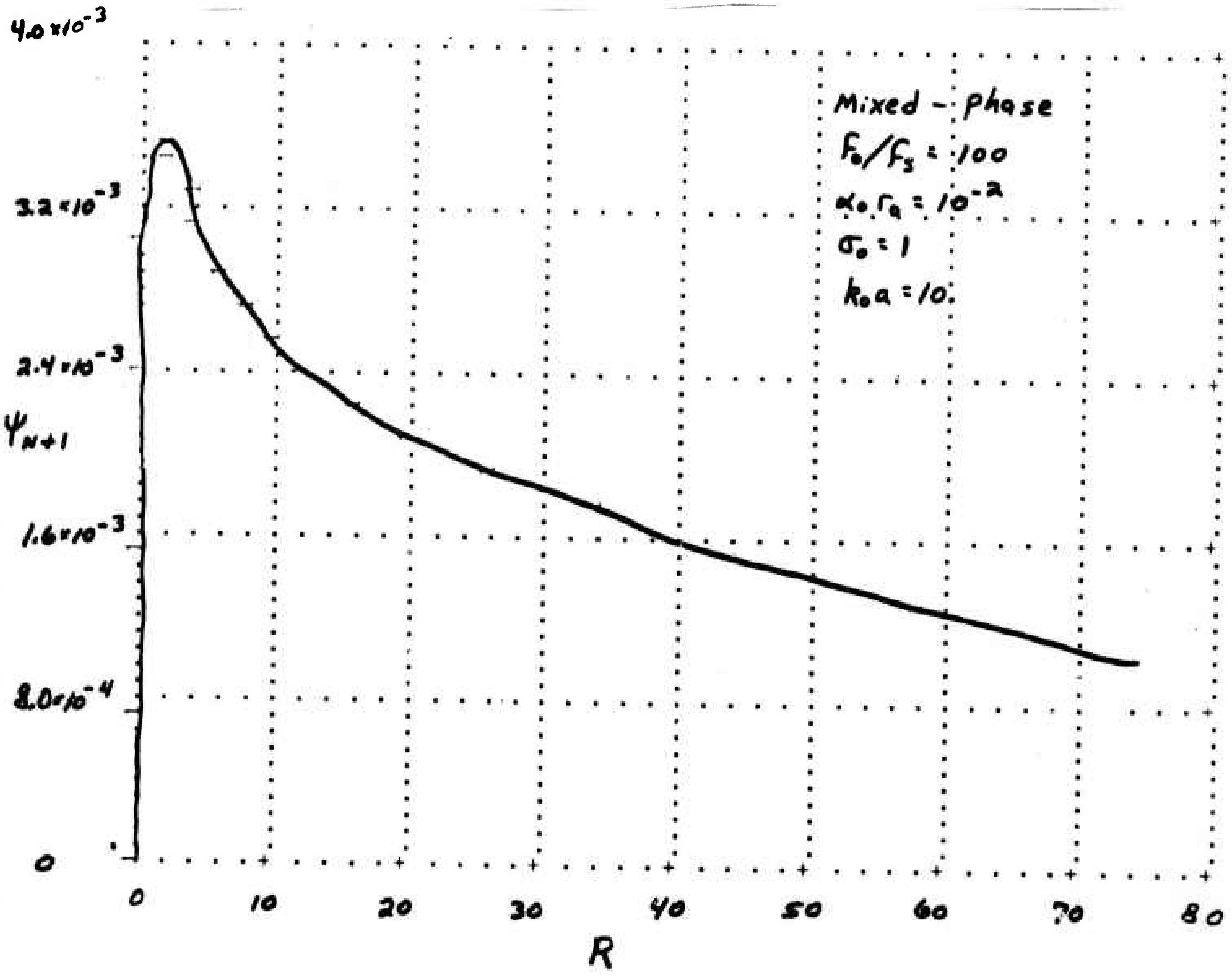
ψ_{NFR+1} - mixed no phase

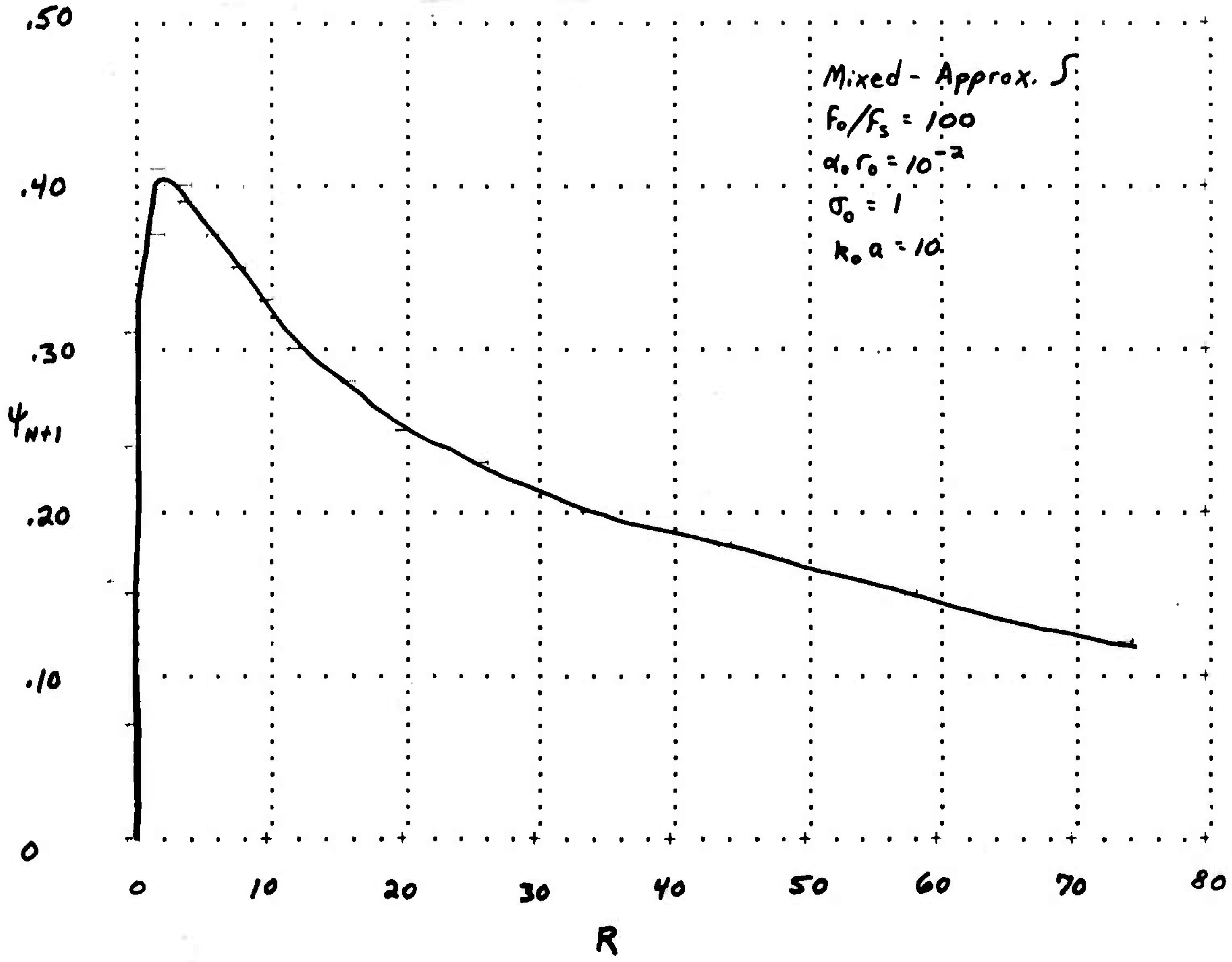
ψ_{NFR+1} - mixed phase

ψ_{NFR+1} - mixed approximate integral form



Mixed – No Phase
$f_o/f_s = 100$
$\alpha_o r_o = 10^{-2}$
$\sigma_o = 1$
$k_o a = 10.$
3.5×10^{-3}
2.8×10^{-3}
2.1×10^{-3}
Ψ_{N+1}
1.4×10^{-3}
7.0×10^{-4}
0
0
10
20
30
40
50
60
70
80
R


4.0×10^-3
3.2×10^-3
2.4×10^-3
1.6×10^-3
8.0×10^-4
0
ψ_{N+1}
0
10
20
30
40
50
60
70
80
R
Mixed - phase
$F_0/F_s = 100$
$\alpha_0 r_a = 10^{-2}$
$\sigma_0 = 1$
$k_0 a = 10.$



.50
.40
.30
.20
.10
0
ψ_{N+1}
0
10
20
30
40
50
60
70
80
R
Mixed - Approx. ∫
$f_0/f_s = 100$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 1$
$k_0 a = 10$

Figure 11

(pages F49 to F51)

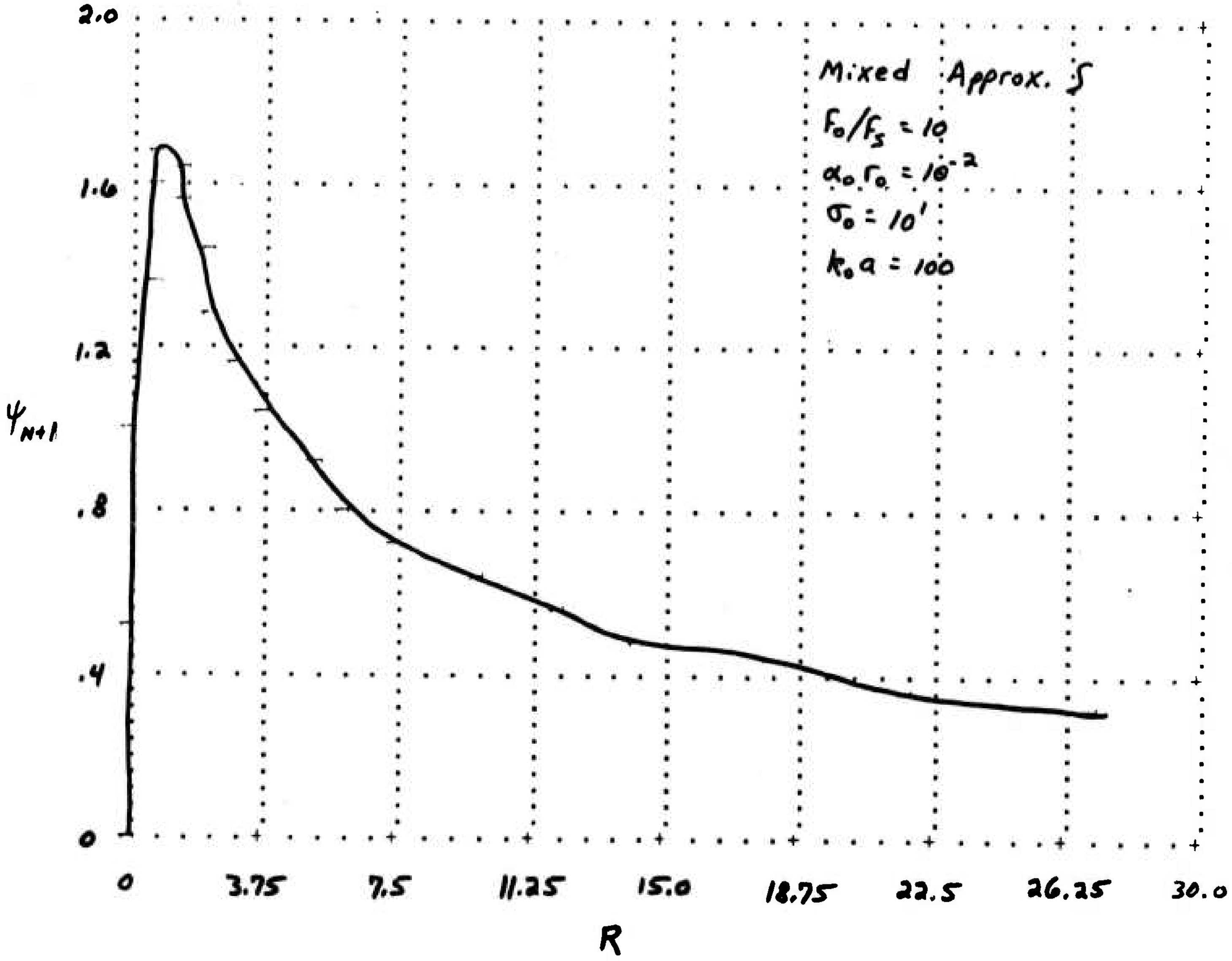
Mixed approximate integral case

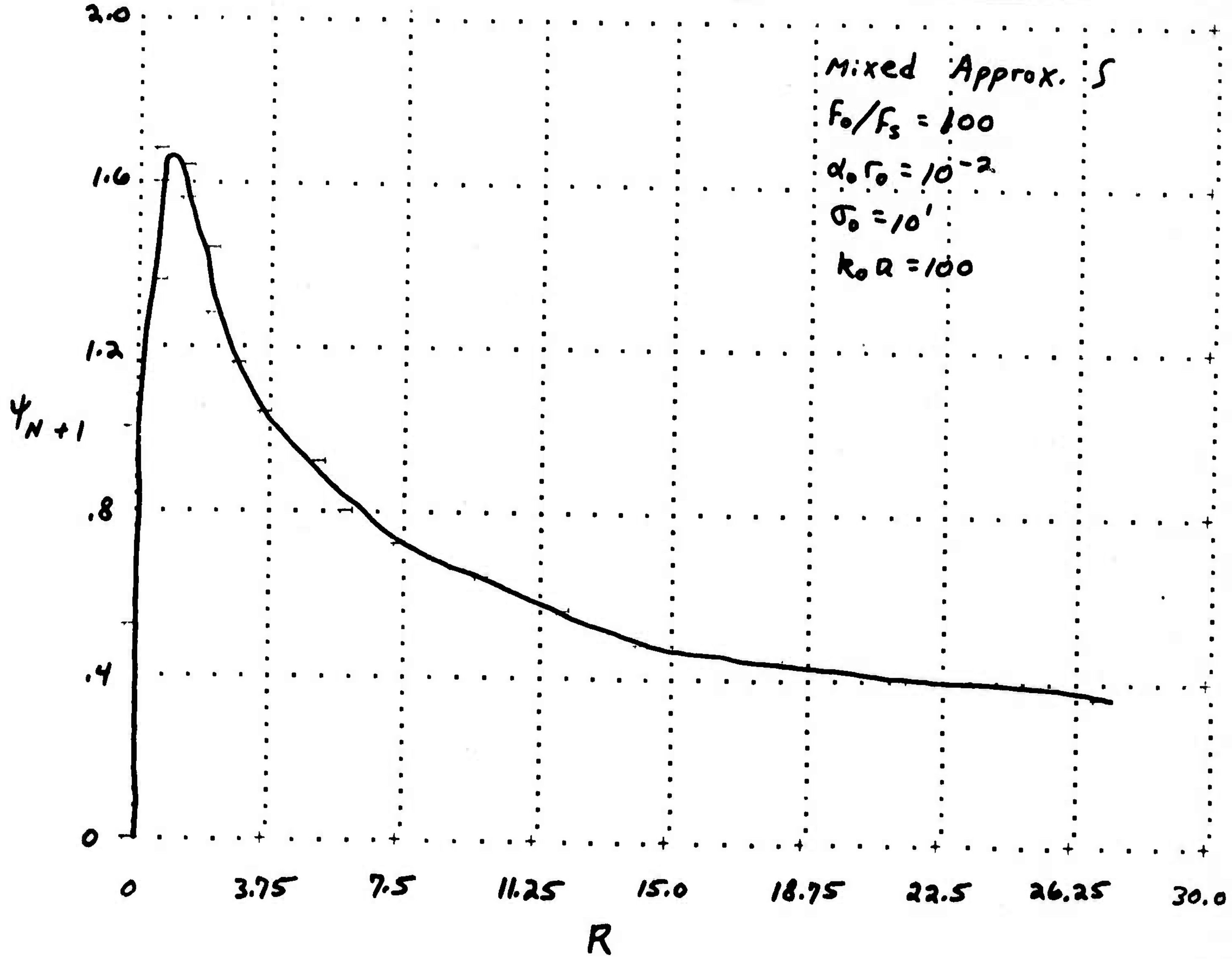
$\alpha_o r_o = 10^{-2}$

$\sigma_o = 10$

$f_o/f_s = 10, 100, 1000$

ψ_{NFR+1}


Mixed Approx. 5
$F_0/F_S = 10$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 10^1$
$k_0 a = 100$
2.0
1.6
1.2
.8
.4
0
ψ_{N+1}
0
3.75
7.5
11.25
15.0
18.75
22.5
26.25
30.0
R


2.0
1.6
1.2
.8
.4
0
ψ_{N+1}
0
3.75
7.5
11.25
15.0
18.75
22.5
26.25
30.0
R
Mixed Approx. 5
$F_0/F_s = 100$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 10^1$
$k_0 a = 100$

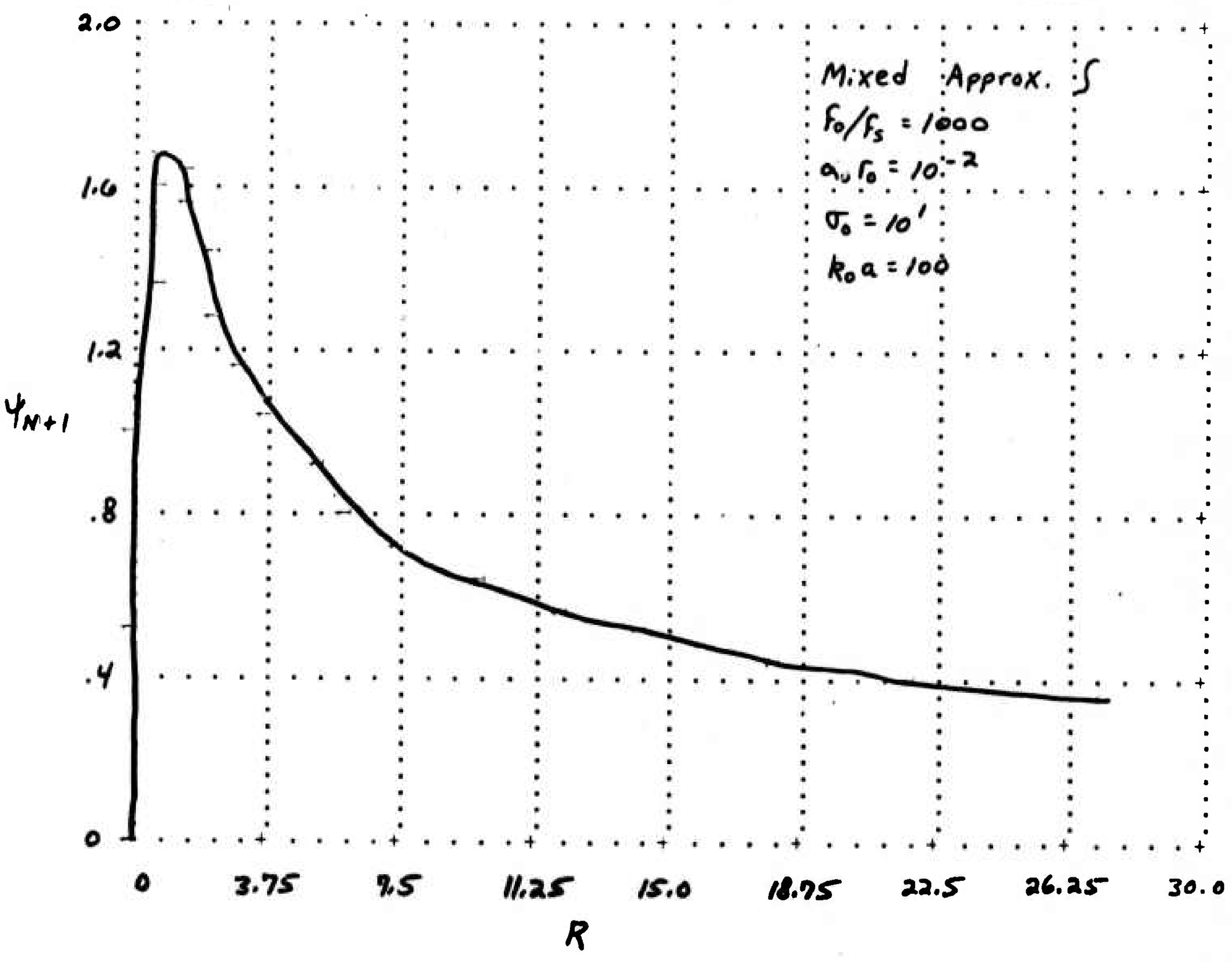

Mixed Approx. ∫
$\rho_0/\rho_s = 1000$
$a_0 r_0 = 10^{-2}$
$\sigma_0 = 10^1$
$k_0 a = 100$
ψ_{N+1}
2.0
1.6
1.2
.8
.4
0
0
3.75
7.5
11.25
15.0
18.75
22.5
26.25
30.0
R

Figure 12

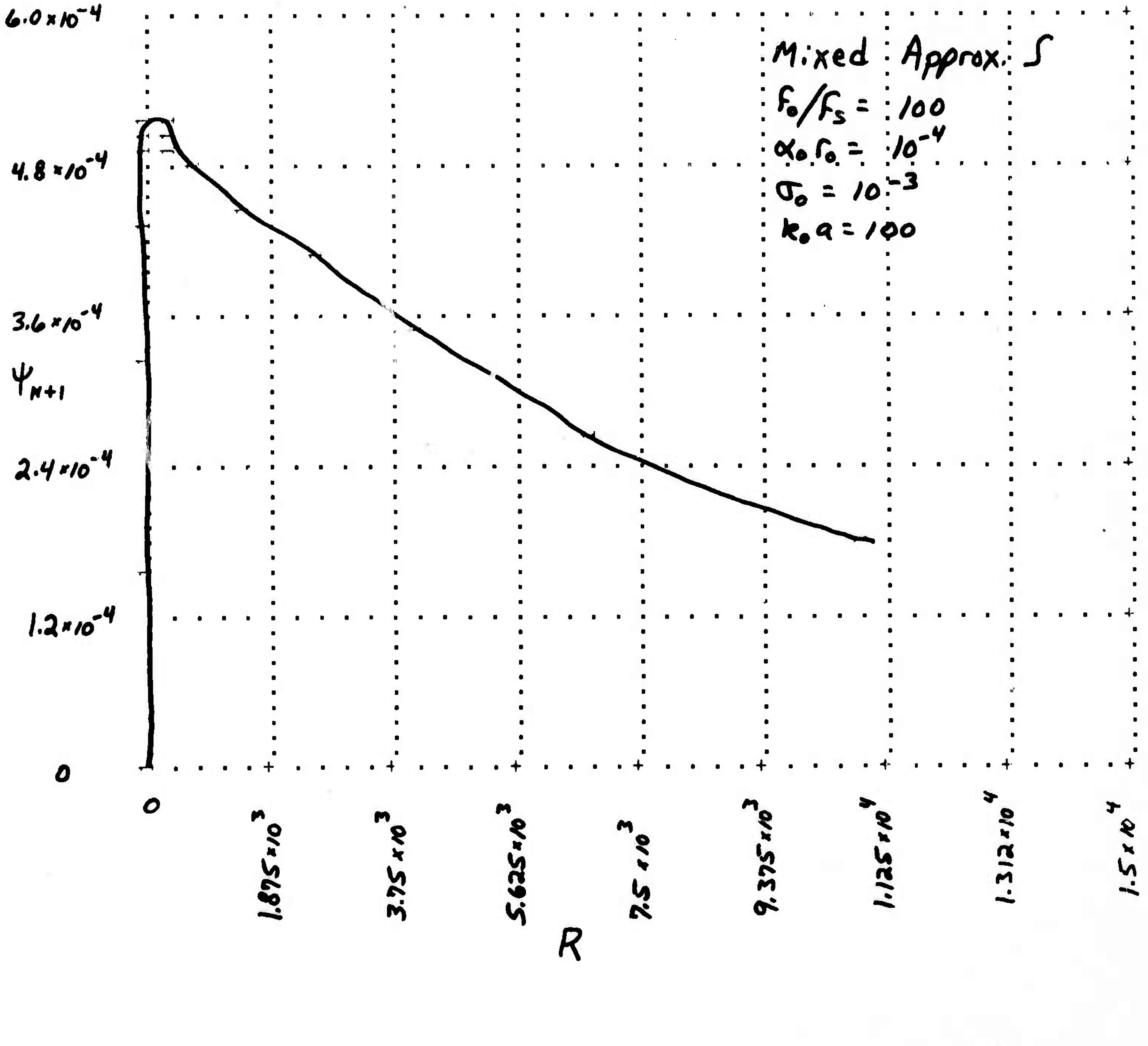
(pagesF53 to F56)

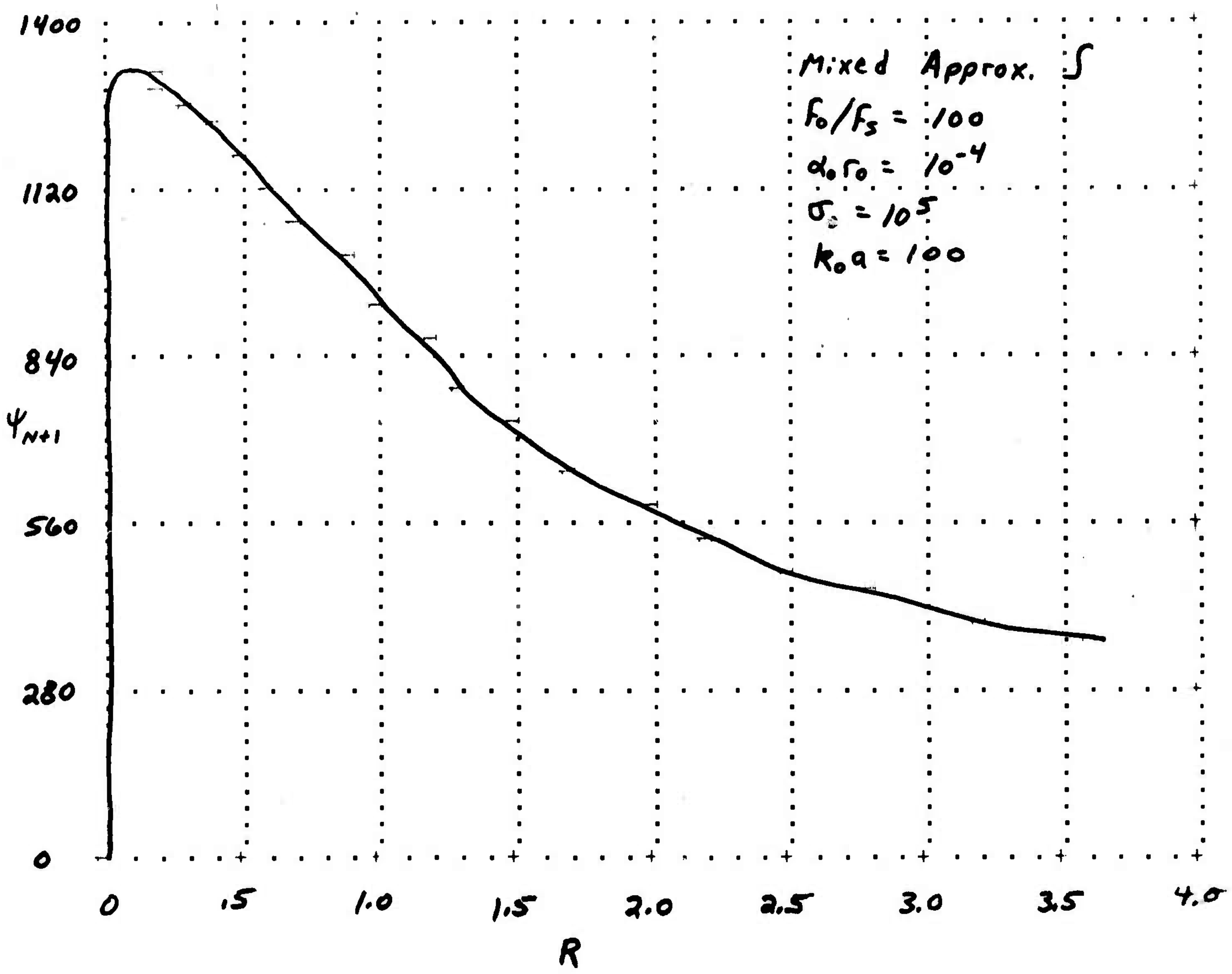
Mixed approximate integral case

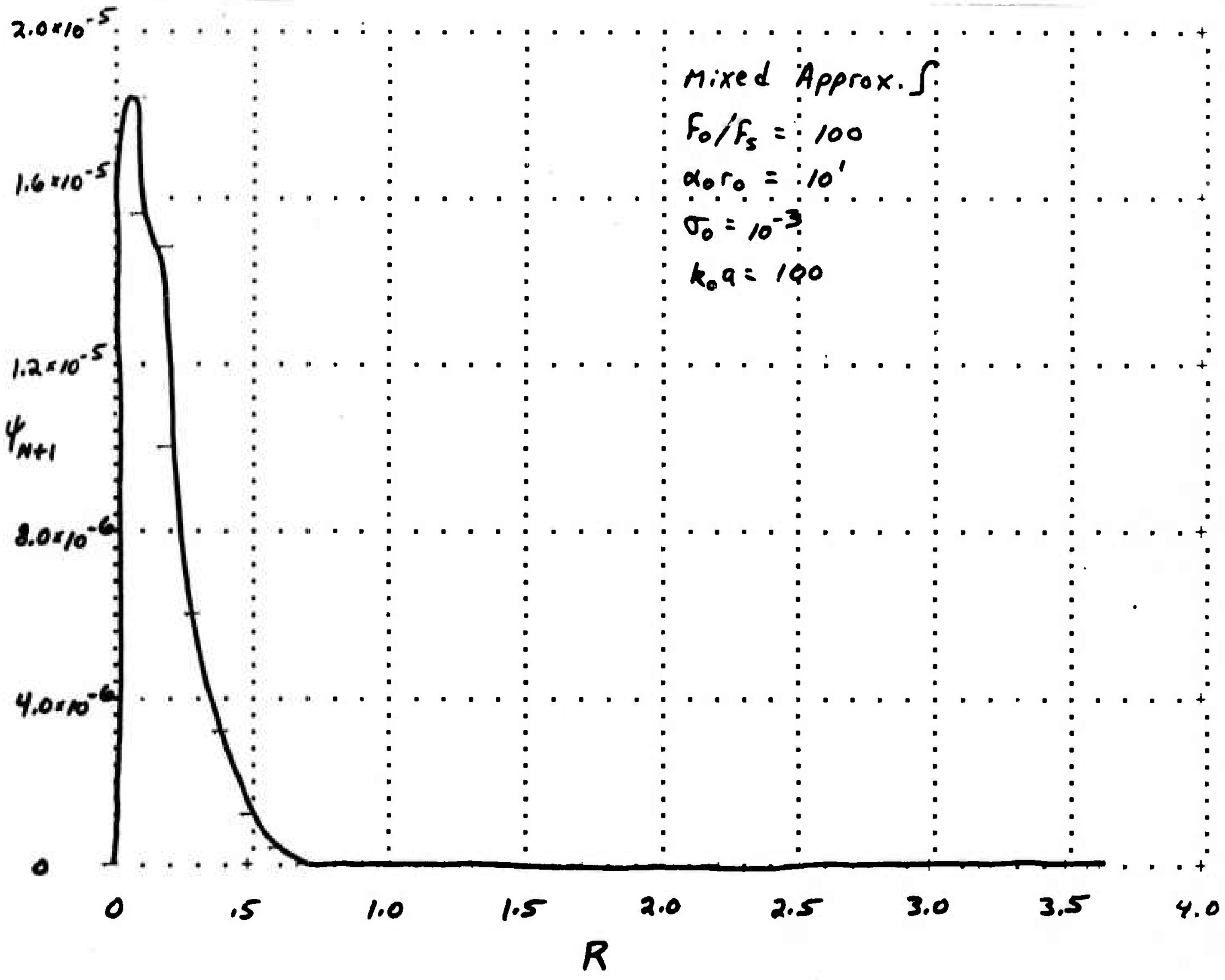
$\alpha_o r_o = 10^{-4}, 10^{1}$

$\sigma_o = 10^{-3}, 10^{5}$

ψ_{NFR+1}


6.0×10⁻⁴
4.8×10⁻⁴
3.6×10⁻⁴
2.4×10⁻⁴
1.2×10⁻⁴
0
ψ_{N+1}
Mixed Approx. S
$f_0/f_s = 100$
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 10^{-3}$
$k_0 a = 100$
0
1.875×10³
3.75×10³
5.625×10³
7.5×10³
9.375×10³
1.125×10⁴
1.312×10⁴
1.5×10⁴
R


1400
1120
840
560
280
0
ψ_{N+1}
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
R
Mixed Approx. ∫
$f_0/f_s = 100$
$\alpha_0 r_0 = 10^{-4}$
$\sigma_0 = 10^5$
$k_0 a = 100$


2.0×10^-5
1.6×10^-5
1.2×10^-5
8.0×10^-6
4.0×10^-6
0
ψ_{N+1}
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
R
Mixed Approx. ∫
F₀/F_s = 100
α₀r₀ = 10¹
σ₀ = 10^-3
k₀a = 100

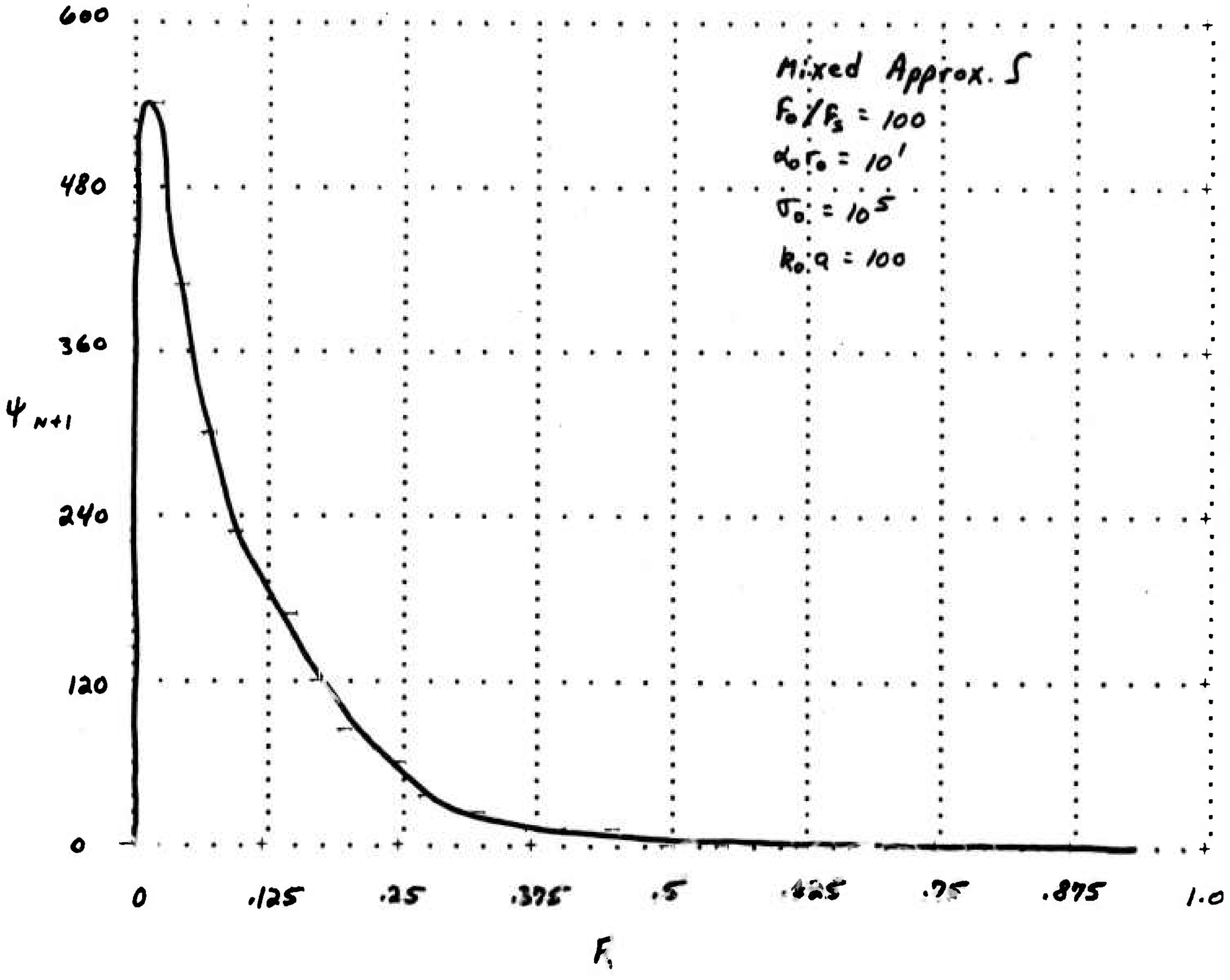

Mixed Approx. 5
$F_0/F_s = 100$
$\alpha_0 r_0 = 10^1$
$\sigma_0 = 10^5$
$k_0 a = 100$
600
480
360
240
120
0
ψ_{N+1}
0
.125
.25
.375
.5
.625
.75
.875
1.0
F_1

Figure 13

(pages F58to F63)

Beam patterns

Mixed approximate integral case

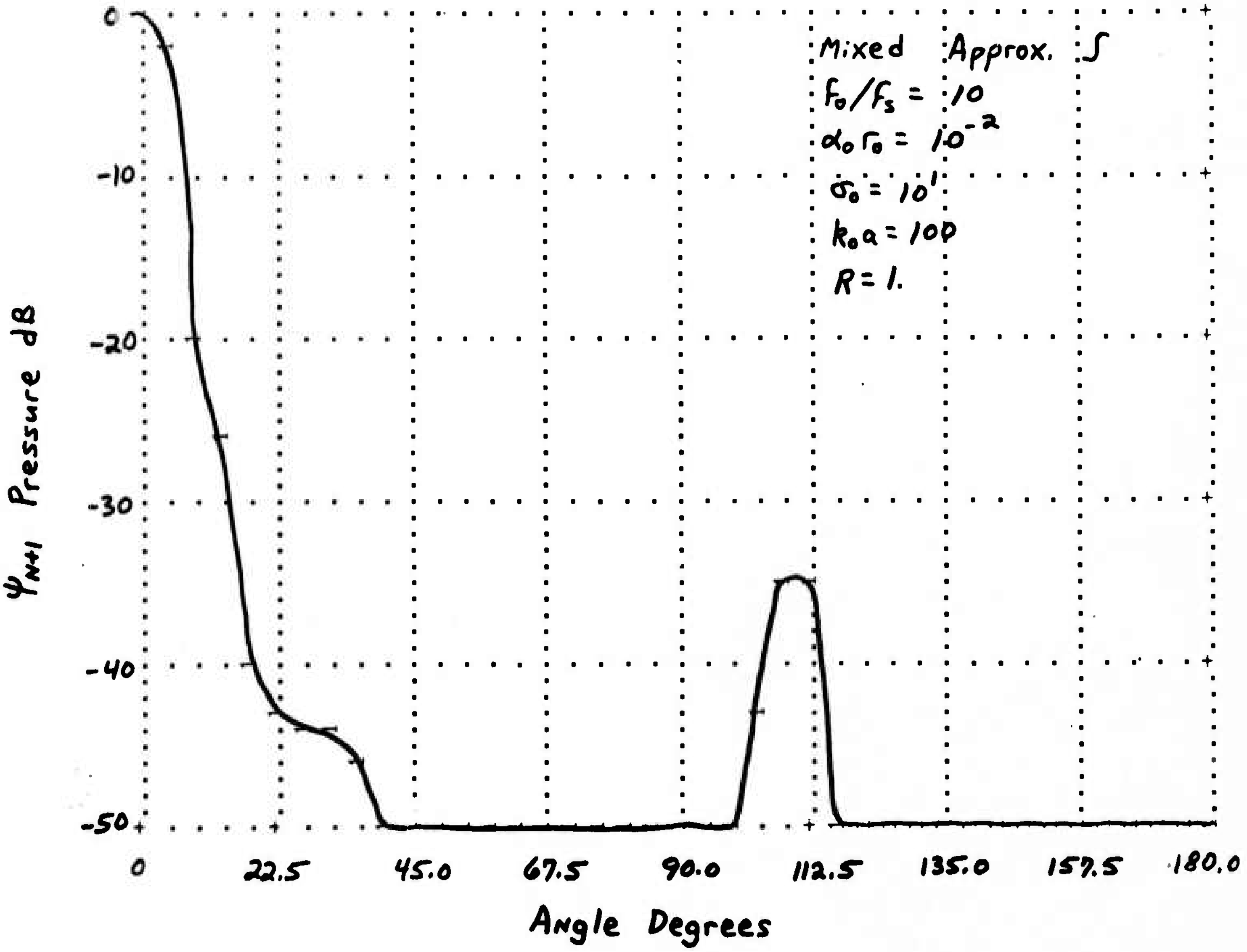
$\alpha_o r_o = 10^{-2}$

$\sigma_o = 10$

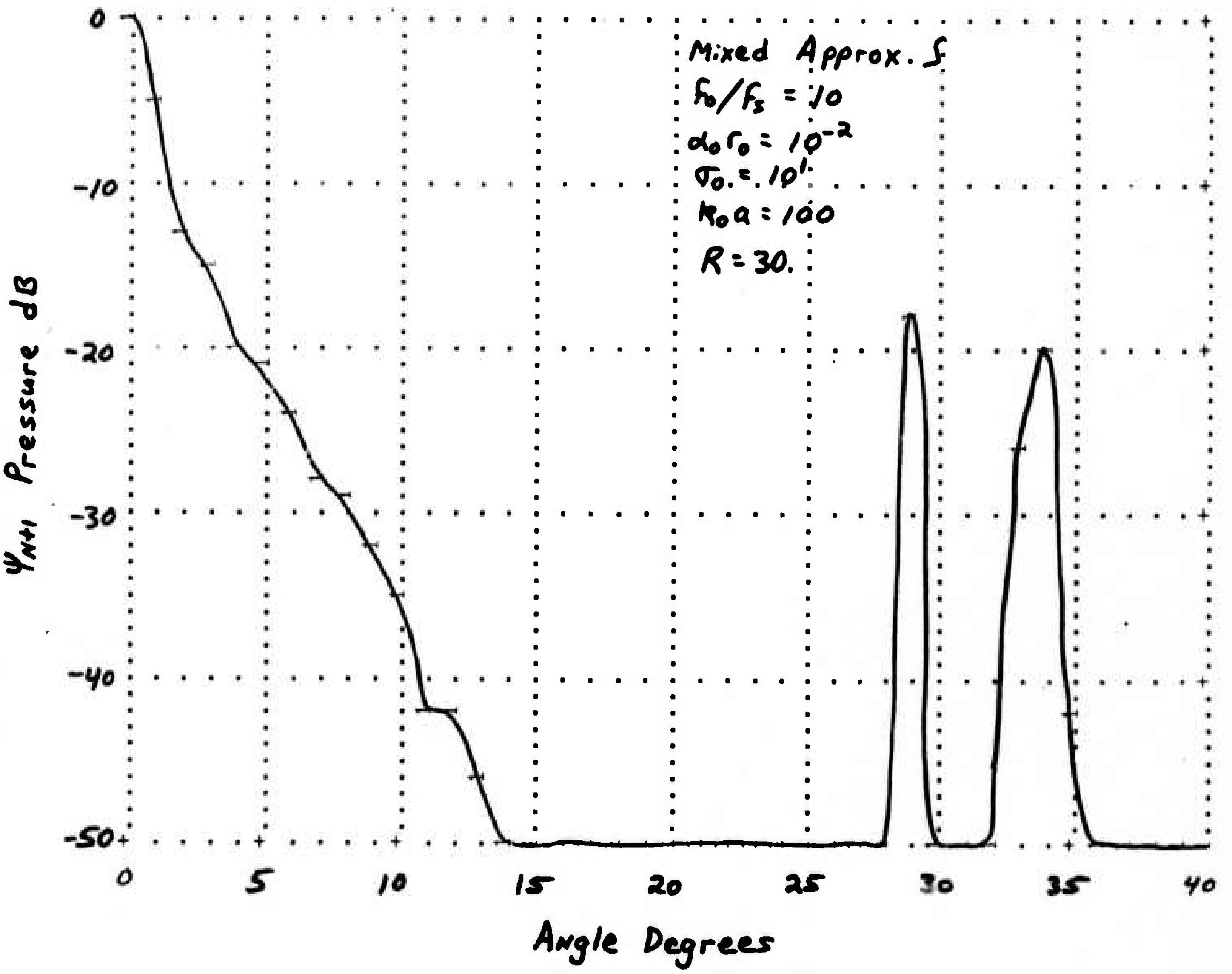
$f_o/f_s = 10, 100, 1000$

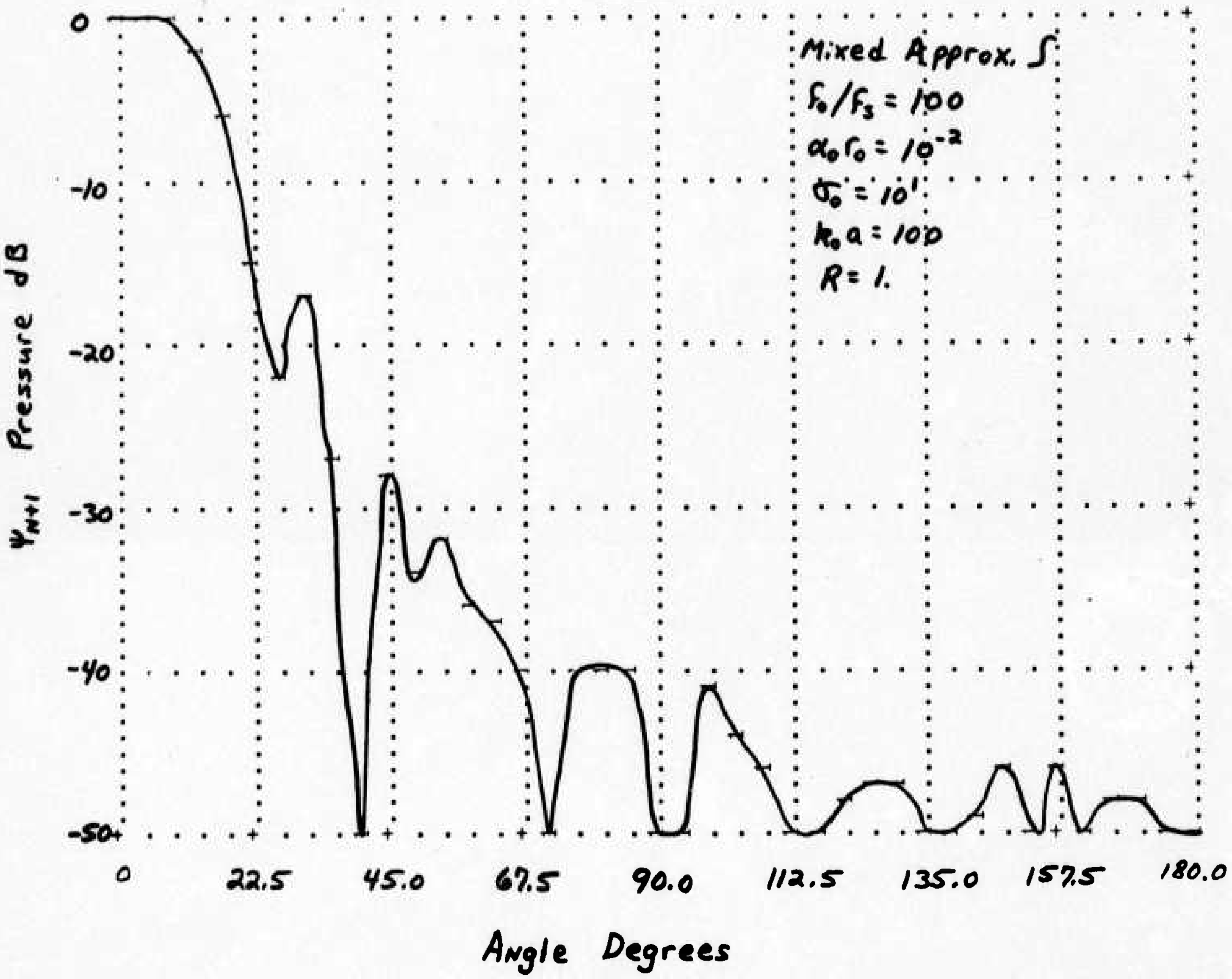
$R = 1, 30$

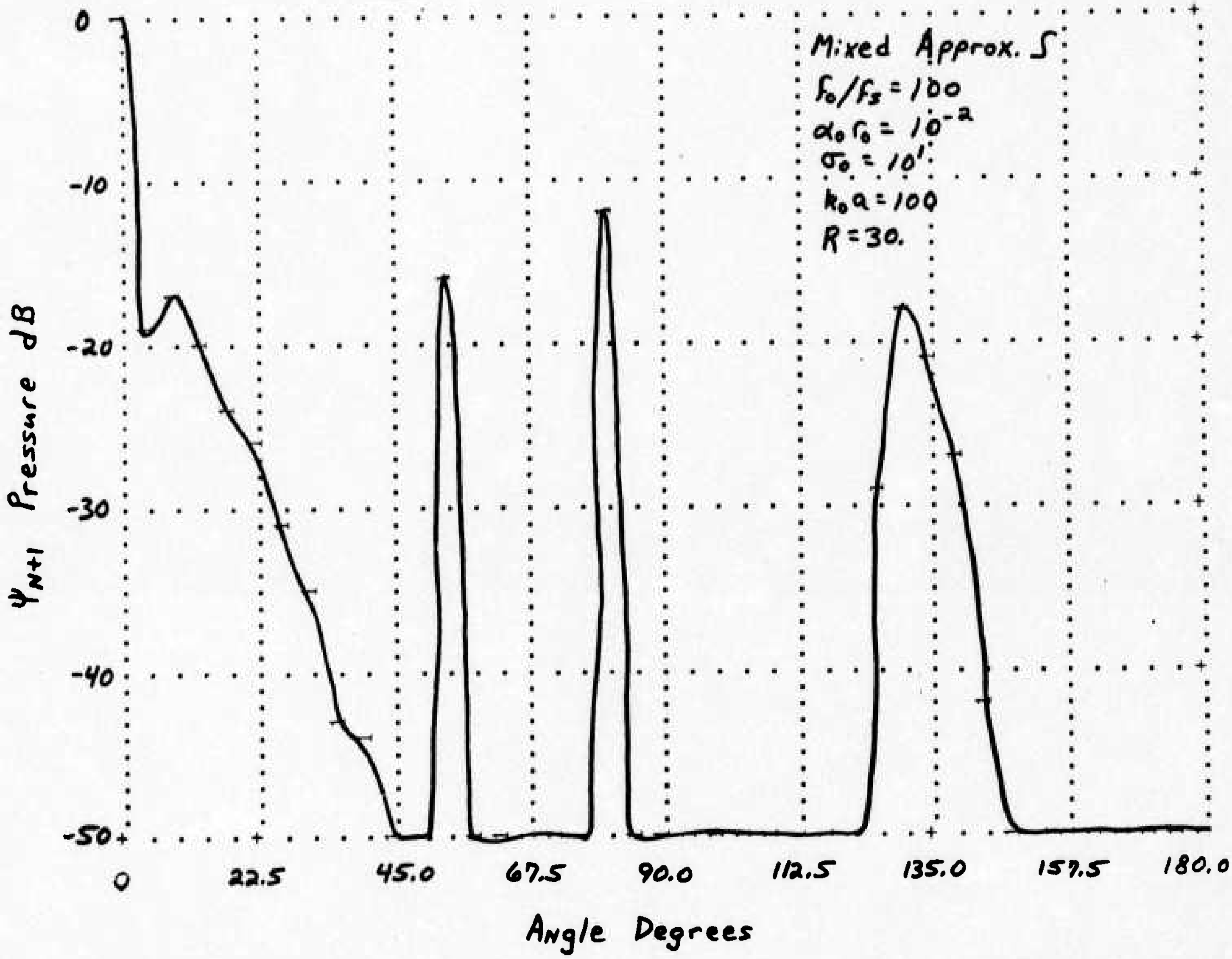
ψ_{NFR+1}

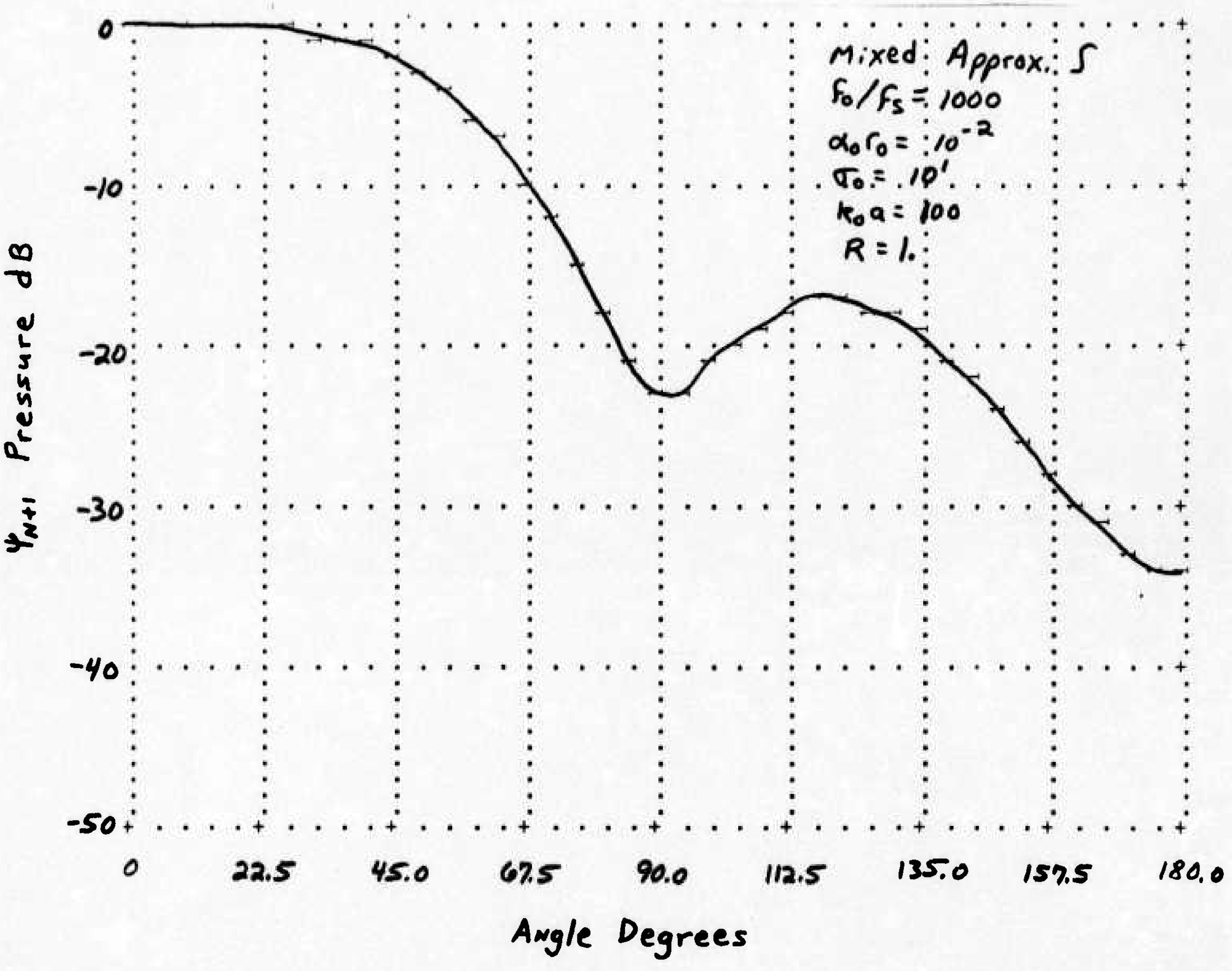


Mixed Approx. ∫
f_o/f_s = 10
$\alpha_o r_o$ = 10^{-2}
σ_o = 10^1
$k_o a$ = 100
R = 1.
0
-10
-20
-30
-40
-50
ψ_{N+1} Pressure dB
0
22.5
45.0
67.5
90.0
112.5
135.0
157.5
180.0
Angle Degrees


Mixed Approx. S
$\rho_0/\rho_s = 10$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 10^1$
$k_0 a = 100$
$R = 30.$
ψ_{N+1} Pressure dB
0
-10
-20
-30
-40
-50
0
5
10
15
20
25
30
35
40
Angle Degrees


Mixed Approx. ∫
$f_0/f_s = 100$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 10^1$
$k_0 a = 100$
$R = 1.$
ψ_{N+1} Pressure dB
0
-10
-20
-30
-40
-50
0
22.5
45.0
67.5
90.0
112.5
135.0
157.5
180.0
Angle Degrees


0
-10
-20
-30
-40
-50
Ψ_{N+1} Pressure dB
0
22.5
45.0
67.5
90.0
112.5
135.0
157.5
180.0
Angle Degrees
Mixed Approx. S
$f_0/f_s = 100$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 10^1$
$k_0 a = 100$
$R = 30.$


0
-10
-20
-30
-40
-50
ψ_{N+1} Pressure dB
0
22.5
45.0
67.5
90.0
112.5
135.0
157.5
180.0
Angle Degrees
Mixed Approx. S
$f_0/f_s = 1000$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 10^1$
$k_0 a = 100$
$R = 1.$

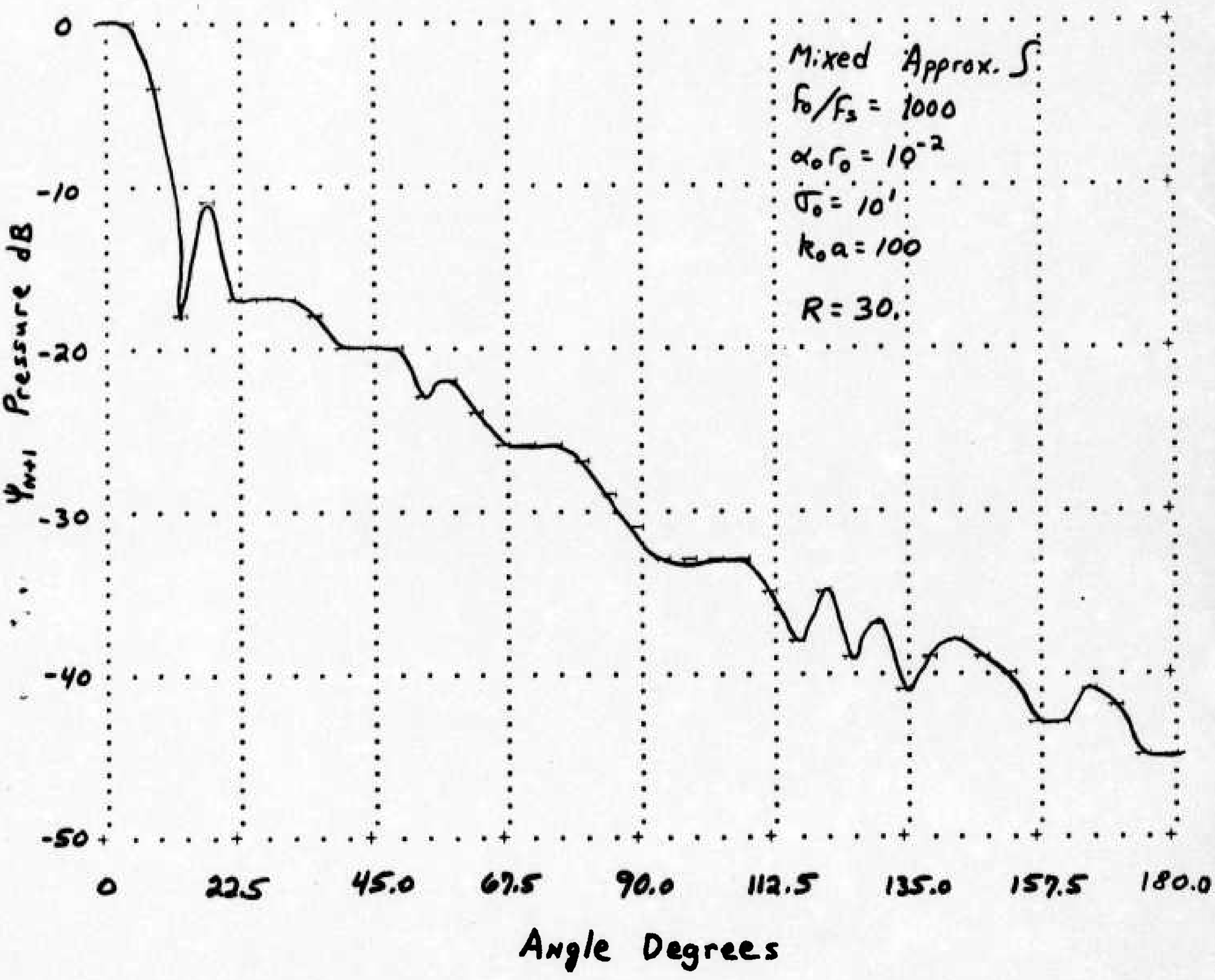

0
-10
-20
-30
-40
-50
Ψ_{N+1} Pressure dB
0
22.5
45.0
67.5
90.0
112.5
135.0
157.5
180.0
Angle Degrees
Mixed Approx. $\int$
$f_0/f_s = 1000$
$\alpha_0 r_0 = 10^{-2}$
$\sigma_0 = 10'$
$k_0 a = 100$
$R = 30.$

Figure 14

(pages F65to F66)

Beam patterns

Mixed approximate integral case

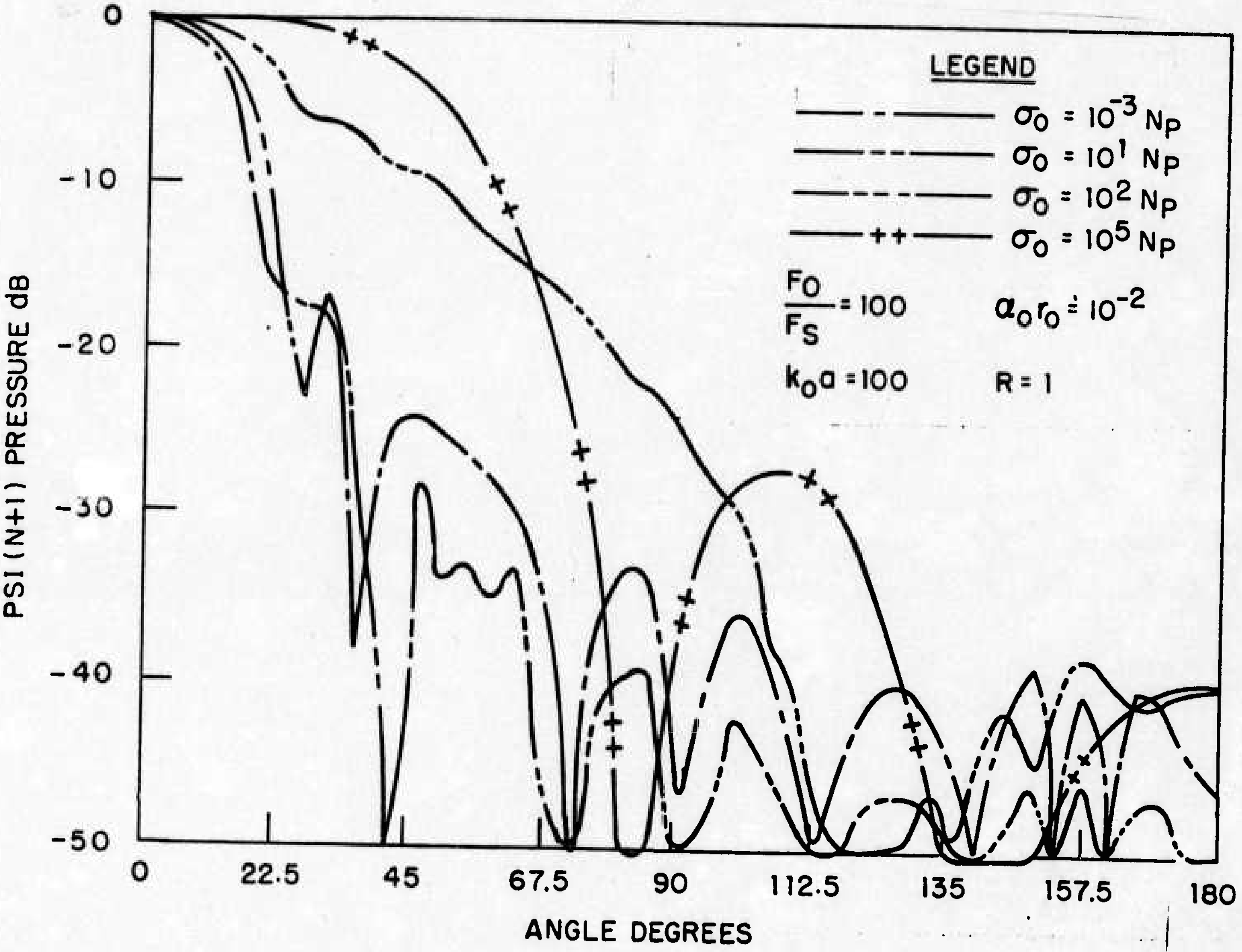
$\alpha_o r_o = 10^{-2}$

$\sigma_o = 10^{-3}, 10^{1}, 10^{2}, 10^{5}$

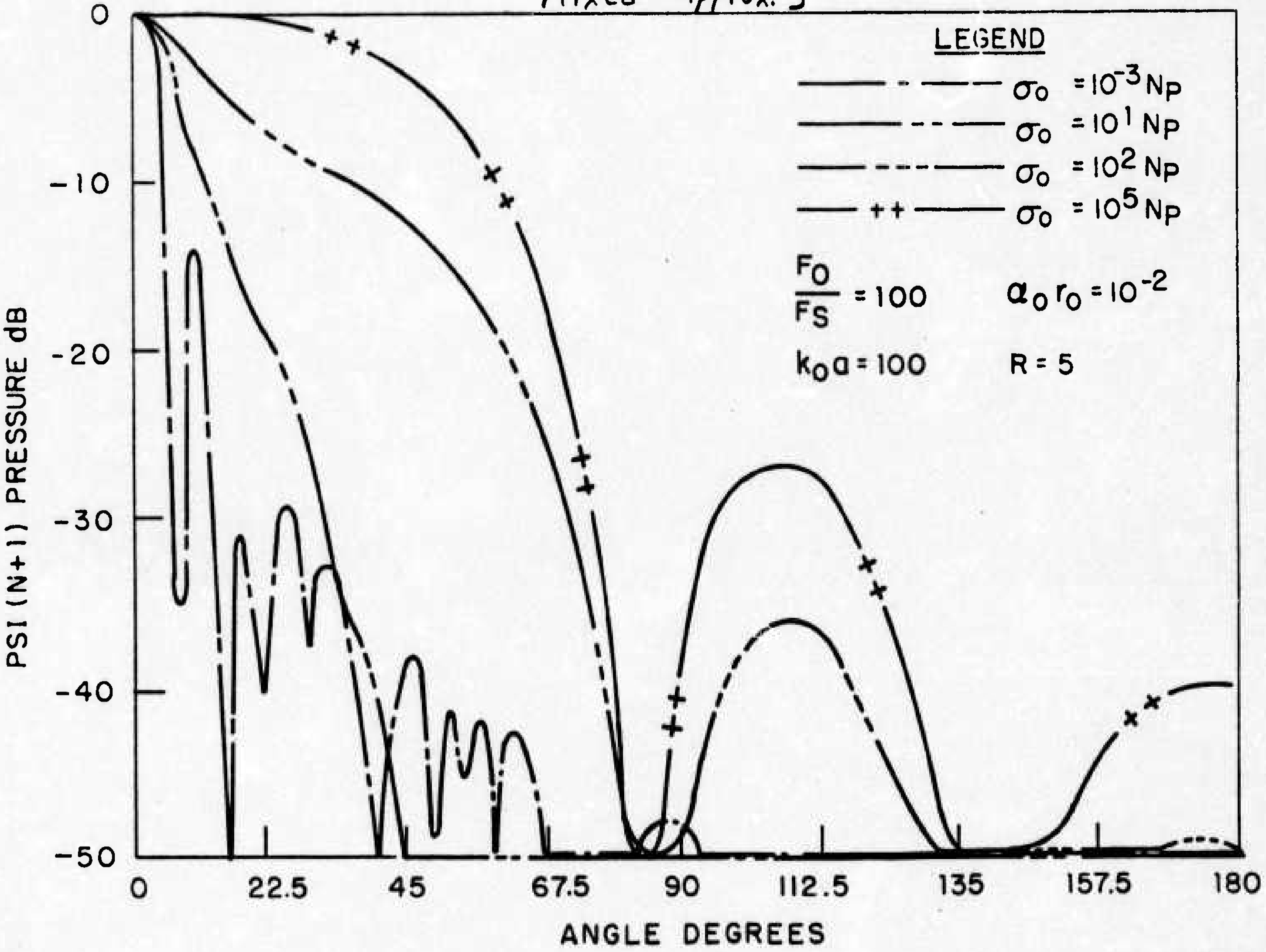
R = 1, 5

ψ_{NFR+1}

Mixed Approx. S



Mixed Approx. S



PROGRAM LISTING

```
      DOUBLE PRECISION R,Y(120),H,DELT,TFINAL,YN(120),TFMD,XMSG,
     1 RLAST
      DIMENSION V(60 ),W(60 ),KD(1),EP(1),KO(120),DT(10,120),
     1 F(120),KS(60 ),AR(5),SG(5),RGOOD(60 ),YGOOD(60 ),YLAST(60 ),
     2 IYN(60 )
      COMMON /BLK1/XP, XD,PS1,PS2,XDB1,IX,IFLAG,M,PSN1,PSN2,
     1 PLO,XMAX,NSC,XT,PAT,TMAX
      DIMENSION XP(22,5),XD(22,5),PS1(22,5),PS2(22,5),XDB1(22,5),
     1 PSN1(22,5),PSN2(22,5),XT(41,5),PAT(41,5)
      COMMON /BLK2/TDPI,DDT,TA,RLAST,RGOOD,YGOOD,YLAST,
     1 IYN,SWK
      DIMENSION PLX(22),PLY(22),PLXT(41),PLYT(41)
      DIMENSION ICHR(5),LBXD(2),LBXP(4),LBYD(4),LBY1(2),LBY2(2),
     1 LBYM(3),LBY3(3),LBY4(3),LBXT(4),LBYT(5)
      COMMON N,FRAT,AUR0,SIG0,F,LPS,KS,NHL,NHL21,NFR,NHL2,LUP,
     1 XMSG,IAP,KSN,KSN1,KSN2,AK0A,IBM,RBM
 1000 FORMAT ( )
 1010 FORMAT (//5H LUP=,I3,2X,5H LPS=,I3,2X,5H NFR=,I3,2X,5H NHL=,
     1 I3,2X,5H NHH=,I3,2X,5H NAR=,I3,2X,5H NSG=,I3,/5H LAP=,I3,
     2 2X,5H NSC=,I3,2X,5H IBM=,I3)
 1020 FORMAT (5H NOC=,I3)
 1030 FORMAT (5H PS0=,E10.4,2X,5H LPP=,I3)
 1040 FORMAT (6H XMAX=,E10.4)
 1050 FORMAT (6H AUR0=,5(E10.4,3X))
 1060 FORMAT (6H SIG0=,5(E10.4,3X))
 1070 FORMAT (6H SALN=,E10.4,3X,6H PRES=,E10.4,3X,6H TEMC=,F6.1,3X,
     1 5H ITA=,I3,3X,/5H SWK=,E10.4,3X,6H K0*A=,E10.4)
 1080 FORMAT (5H IAP=,I3)
 1090 FORMAT (5H RBM=,E10.4,3X,5H TMAX=,F7.2)
 2010 FORMAT (4H EP=,E10.4,3X,7H HMINA=,E10.4)
 2020 FORMAT (25H ERROR IN CONTROL PARAMETERS)
 2030 FORMAT (40H APPROX. EQNS. NOT VALID FOR DIFF. FREQ.)
 2040 FORMAT (41H APPROX. EQNS. ONLY VALID FOR MIXED CASES)
      READ 1000,NOC
      PRINT 1020,NOC
      DO 400 I=1,NOC
      READ 1000,LUP,LPS,NFR,NHL,NHH,NAR,NSG,LAP,NSC,IBM
      PRINT 1010,LUP,LPS,NFR,NHL,NHH,NAR,NSG,LAP,NSC,IBM
C     LUP=1 FOR HARMONIC STUDIES,=2 FOR UP-CONVERSION,=3 FOR DIFF.
C     FREQ.
C     LPS=0 FOR PLANE WAVE, =1 FOR SPHERICAL, =2 FOR COMBINED PHASE
C        =3 FOR COMBINED NO PHASE
C     LAP=1 IF WANT APPROXIMATE EQUATIONS,=0 OTHERWISE
C     NSC=0 FOR NO SIG0 SCALE,=1 FOR SIG0 SCALE
C     IBM=1 FOR BEAM PATTERN,=0 OTHERWISE
C     IBM=1 ONLY FOR LUP=2 AND LAP=1
C     INPUT CAN BE ORGANIZED BETTER
      IF (NSG .GT. 5) NSG=5
      READ 1000,XMAX
      PRINT 1040,XMAX
      READ 1000,(AR(L),L=1,NAR)
      PRINT 1050,(AR(L),L=1,NAR)
      READ 1000,(SG(L),L=1,NSG)
      PRINT 1060,(SG(L),L=1,NSG)
      READ 1000,SALN,PRES,TEMC,ITA,SWK,AKOA
      PRINT 1070,SALN,PRES,TEMC,ITA,SWK,AKOA
C     SALN=SALINITY IN PARTS PER 1000, =0. FOR FRESH WATER
C     PRES=PRESSURE IN ATMOSPHERES
C     TEMC=TEMPERATURE IN DEGREES CENTIGRADE
C     ITA=1 FOR MARSH+SCHULKIN FORM, =2 FOR RUSSIAN FORM
C     SWK=SEA WATER K, AKOA=KO*A
      TDPD=0.69145*SALN
      DDT=0.538E-3*(1.0-6.54E-4*PRES)
      IF (ITA .EQ. 1) TA=7.267E-12*10.0**(1520.0/(TEMC+273.0))
      IF (ITA .EQ. 2) TA=(1.0E-3/755.0)/(TEMC+273.0)*10.0**(934.0/
     1 (TEMC+273.0))
      IF (LAP .NE. 1) GO TO 118
      READ 1000,IAP
      PRINT 1080,IAP
C     IAP=1 FOR 1ST APPROX. FORM,=2 FOR 2ND APPROX. FORM
      IF (IBM .NE. 1) GO TO 205
      READ 1000,RBM,TMAX
      PRINT 1090,RBM,TMAX
  205 CONTINUE
      IF (LPS .GE. 2) GO TO 204
      PRINT 2040
      GO TO 999
  204 IF (LUP .NE. 3) GO TO 118
      PRINT 2030
      GO TO 999
  118 CONTINUE
      IF (LUP .NE. 2) GO TO 101
      READ 1000,PSO,LPP
      PRINT 1030,PSO,LPP
C     LPP=1 FOR ONLY N-1 CURVE,=2 FOR ONLY N+1 CURVE,=3 FOR BOTH
  101 CONTINUE
      ICHR(1)='1'
      ICHR(2)='2'
      ICHR(3)='3'
      ICHR(4)='4'
      ICHR(5)='5'
      LBXD(1)=1
      LBXD(2)='9'
      IF (LPS .NE. 0) GO TO 114
      IF (NSC .EQ. 0) GO TO 201
      LBXP(1)=6
      LBXP(2)='SIG0*R'
      GO TO 115
  201 LBXP(1)=1
      LBXP(2)='R'
      GO TO 115
  114 IF (LPS .NE. 1) GO TO 116
      IF (NSC .EQ. 0) GO TO 202
      LBXP(1)=10
      LBXP(2)='SIG0*L'
      LBXP(3)='N(R)'
```

```
2060          GO TO 115
2070      202 LBXP(1)=5
2080          LBXP(2)='LN(R)'
2090          GO TO 115
2100      116 IF (NSC .EQ. 0) GO TO 203
2110          LBXP(1)=15
2120          LBXP(2)='SIGO*A'
2130          LBXP(3)='RCSINH'
2140          LBXP(4)='(R)'
2150          GO TO 115
2160      203 LBXP(1)=10
2170          LBXP(2)='ARCSIN'
2180          LBXP(3)='H(R)'
2190      115 CONTINUE
2200          LBYD(1)=13
2210          LBYD(2)='EXTRA '
2220          LBYD(3)='DB LOS'
2230          LBYD(4)='S'
2240          LBY1(1)=4
2250          LBY1(2)='PSI1'
2260          LBY2(1)=4
2270          LBY2(2)='PSI2'
2280          LBYM(1)=11
2290          LBYM(2)='DIFF. '
2300          LBYM(3)='FREQ.'
2310          LBY3(1)=8
2320          LBY3(2)='PSI(N-'
2330          LBY3(3)='1)'
2340          LBY4(1)=8
2350          LBY4(2)='PSI(N+'
2360          LBY4(3)='1)'
2370          IF (IBM .NE. 1) GO TO 120
2380          LBXT(1)=13
2390          LBXT(2)='ANGLE '
2400          LBXT(3)='DEGREE'
2410          LBXT(4)='S'
2420          LBYT(1)=20
2430          LBYT(2)='PSI(N+'
2440          LBYT(3)='1) PRE'
2450          LBYT(4)='SSURE '
2460          LBYT(5)='DB'
2470      120 CONTINUE
2480          DO 350 L=1,NAR
2490          AORO=AR(L)
2500          DO 351 M=1,NSG
2510          SIGO=SG(M)
2520          XMSG=XMAX/SIGO
2530          IF (NSC .EQ. 0) XMSG=XMAX
2540          FRAT=1.0/NFR
2550          F2AR=FRAT**2 *AORO
2560          IF (LUP .NE. 2) GO TO 102
2570          AKORO=(AKOA**2)/2.0
2580          FKR2=2.0*FRAT*AKORO
```

```
      THET=0.0
      ST22=(SIN(THET/2.0))**2
  102 CONTINUE
      IF (LAP .NE. 1) GO TO 119
      CALL APPROX
      GO TO 351
  119 CONTINUE
      NHL2=2*NHL
      NHL21=NHL2+1
      IF (NFR .LE. NHL2) GO TO 108
      N=NHL+NHH*NHL21
      KSN=NHL2+1
      DO 109 J=1,N
      NLJ=NHL+1+J
      JMN=MOD(NLJ,NHL21)
      IF (JMN .EQ. 0) JMN=NHL21
      QJN=(NLJ-0.5)/NHL21
      JDN=INT(QJN)
      KR=JDN*NFR+(JMN-NHL-1)
      KS(J)=KR
  109 CONTINUE
      GO TO 110
  108 N=NHH*NFR+NHL
      KSN=NFR
      DO 111 J=1,N
  111 KS(J)=J
  110 CONTINUE
      KSN1=KSN-1
      KSN2=KSN+1
      NEQ=2*N
      DO 105 K=1,N
      V(K)=0.
      W(K)=0.
  105 CONTINUE
      IF (LUP .NE. 2) GO TO 103
      V(KSN )=0.0
      W(KSN )=1.0
      GO TO 112
  103 IF (LUP .EQ. 1) W(1)=1.0
      IF (LUP .NE. 3) GO TO 112
      W(KSN)=0.5
      W(KSN1)=0.5
C     ASSUME PUMP FREQS. IN NFR-1,NFR AND DIFF. FREQ. IN 1
  112 CONTINUE
      KD(1)=1
      NHM=0
      IF (LPS .NE. 0) GO TO 106
      P=0.0D0
      RLO=0.0
      IF (LUP .EQ. 2) W(1)=PS0
      TFINAL=XMSG
      EP(1)=1.0E-3
      H=TFINAL*1.0D-2
      HMINA=TFINAL*1.0E-4
      HMAXA=TFINAL/2.0
      DELT=TFINAL/20.0
      MXSTEP=20
      TFMD=TFINAL-HMINA
      GO TO 107
  106 IF (LPS .GE. 2) GO TO 117
      R=1.0D0
      RLO=1.0
      IF (LUP .EQ. 2) W(1)=PS0*EXP(-FRAT**2 *A0R0)
      TFINAL=DEXP(XMSG)
      EP(1)=1.0E-3
      H=TFINAL*1.0D-2
      HMINA=TFINAL*1.0E-4
      HMAXA=TFINAL/2.0
      DELT=DEXP(XMSG/20.0D0)-1.0D0
      MXSTEP=20
      TFMD=TFINAL-HMINA
      GO TO 107
  117 R=1.0D-3
      RLO=0.0
      IF (LUP .EQ. 2) W(1)=PS0*EXP(-FRAT**2*A0R0)
      TFINAL=DEXP(XMSG)/2.0D0
      EP(1)=1.0E-3
      H=TFINAL*1.0D-2
      HMINA=TFINAL*1.0E-4
      HMAXA=TFINAL/2.0
      DELT=0.5D0*(DEXP(XMSG/20.0D0)-1.0D0)
      MXSTEP=20
      TFMD=TFINAL-HMINA
  107 CONTINUE
      DO 104 KI=1,N
      Y(2*KI-1)=V(KI)
      Y(2*KI)  =W(KI)
      IYN(KI)=0
  104 CONTINUE
      IX=0
      CALL VODQ(NEQ,R,Y,F,KD,EP,IFLAG,H,HMINA,HMAXA,DELT,TFINAL,
     1 MXSTEP,KSTEP,KEMAX,EMAX,KQ,YN,DT)
      GO TO 6
    4 CALL VODQ1
    6 GO TO (10,10,30,40,50,60,70,80),IFLAG
   10 CONTINUE
      IF (LUP .NE. 2) GO TO 113
      PEFR=PS0*EXP(-F2AR*R)
      FRST=FKR2*R*ST22
      Y(1)=PEFR*SIN(FRST)
      Y(2)=PEFR*COS(FRST)
  113 CONTINUE
      CALL DERIV(R,Y)
      GO TO 4
   30 IF (R .GE. TFMD  ) GO TO 200
      CALL OUTPUT(R,Y)
      IF (IX .GE. 22) GO TO 300
      IF (LPS .EQ. 0) GO TO 4
      IF (LPS .EQ. 1) DELT=R◆(DEXP(XMSG/20.0D0)-1.0D0)
      IF (LPS .GE. 2) DELT=(R+0.5D0)◆(DEXP(XMSG/20.0D0)-1.0D0)
      GO TO 4
   40 GO TO 200
   50 CALL OUTPUT(R,Y)
      GO TO 4
   60 EP(1)=32.◆EMAX◆EP(1)
      GO TO 4
   70 HMINA=HMINA/10.0
      PRINT 2010,EP(1),HMINA
      TFMD=TFINAL-HMINA
      NHM=NHM+1
      IF (NHM .LE. 3) GO TO 4
      GO TO 300
   80 PRINT 2020
      GO TO 999
  200 CALL OUTPUT(R,Y)
  300 CONTINUE
  351 CONTINUE
      IF (IBM .NE. 1) GO TO 360
      CALL TSETUP(0.0,TMAX,-50.0,0.0,1HF,1HG,LBXT,LBYT)
      DO 361 M=1,NSG
      DO 362 K=1,41
      PLXT(K)=XT(K,M)
  362 PLYT(K)=PAT(K,M)
      IF (M .EQ. NSG) GO TO 363
      CALL TPLOT(.FALSE.,ICHR(M),41,PLXT,PLYT)
      GO TO 361
  363 CALL TPLOT(.TRUE.,ICHR(M),41,PLXT,PLYT)
  361 CONTINUE
      GO TO 350
  360 IF (LUP .NE. 1) GO TO 315
      CALL TSETUP(0.0,XMAX,0.0,1.0,1HE,1HG,LBXP,LBY1)
      DO 301 M=1,NSG
      DO 302 K=1,IX
      PLX(K)=XP(K,M)
  302 PLY(K)=PS1(K,M)
      IF (M .EQ. NSG) GO TO 303
      CALL TPLOT(.FALSE.,ICHR(M),IX,PLX,PLY)
      GO TO 301
  303 CALL TPLOT(.TRUE.,ICHR(M),IX,PLX,PLY)
  301 CONTINUE
      IF (LAP .EQ. 1) GO TO 317
      DO 304 M=1,NSG
      DO 305 K=1,IX
      PLX(K)=XP(K,M)
  305 PLY(K)=PS2(K,M)
      IF (M .GT. 1) GO TO 306
      CALL TSCALE(PLY,IX,VLOW,VHIGH,.TRUE.,1)
      GO TO 304
  306 CALL TSCALE(PLY,IX,VLOW,VHIGH,.FALSE.,1)
  304 CONTINUE
      CALL TSETUP(0.0,XMAX,0.0,VHIGH,1HE,1HG,LBXP,LBY2)
      DO 307 M=1,NSG
      DO 314 K=1,IX
      PLX(K)=XP(K,M)
  314 PLY(K)=PS2(K,M)
      IF (M .EQ. NSG) GO TO 308
      CALL TPLOT(.FALSE.,ICHR(M),IX,PLX,PLY)
      GO TO 307
  308 CALL TPLOT(.TRUE.,ICHR(M),IX,PLX,PLY)
  307 CONTINUE
  317 DO 312 M=1,NSG
      DO 310 K=1,IX
      PLX(K)=XD(K,M)
  310 PLY(K)=XDB1(K,M)
      PLY(1)=PLY(2)
      IF (M .GT. 1) GO TO 311
      CALL TSCALE(PLY,IX,VLOW,VHIGH,.TRUE.,1)
      CALL TSCALE(PLX,IX,RLOW,RHIGH,.TRUE.,0)
      GO TO 312
  311 CALL TSCALE(PLY,IX,VLOW,VHIGH,.FALSE.,1)
      CALL TSCALE(PLX,IX,RLOW,RHIGH,.FALSE.,0)
  312 CONTINUE
      CALL TSETUP(RLO,RHIGH,VLOW,VHIGH,1HE,1HG,LBXD,LBYD)
      DO 309 M=1,NSG
      DO 316 K=1,IX
      PLX(K)=XD(K,M)
  316 PLY(K)=XDB1(K,M)
      PLY(1)=PLY(2)
      IF (M .EQ. NSG) GO TO 313
      CALL TPLOT(.FALSE.,ICHR(M),IX,PLX,PLY)
      GO TO 309
  313 CALL TPLOT(.TRUE.,ICHR(M),IX,PLX,PLY)
  309 CONTINUE
      GO TO 350
  315 IF (LUP .NE. 2) GO TO 320
      IF (LPP .EQ. 2) GO TO 337
      DO 331 M=1,NSG
      DO 332 K=1,IX
      PLX(K)=XD(K,M)
  332 PLY(K)=PSN1(K,M)
      IF (M .GT. 1) GO TO 333
      CALL TSCALE(PLY,IX,VLOW,VHIGH,.TRUE.,1)
      CALL TSCALE(PLX,IX,RLOW,RHIGH,.TRUE.,0)
      GO TO 331
  333 CALL TSCALE(PLY,IX,VLOW,VHIGH,.FALSE.,1)
      CALL TSCALE(PLX,IX,RLOW,RHIGH,.FALSE.,0)
  331 CONTINUE
      CALL TSETUP(RLO,RHIGH,0.0,VHIGH,1HE,1HG,LBXD,LBY3)
      DO 334 M=1,NSG
      DO 335 K=1,IX
      PLX(K)=XD(K,M)
```

```
4710   335 PLY(K)=PSN1(K,M)
4720       IF (M .EQ. NSG) GO TO 336
4730       CALL TPLOT(.FALSE.,ICHR(M),IX,PLX,PLY)
4740       GO TO 334
4750   336 CALL TPLOT(.TRUE.,ICHR(M),IX,PLX,PLY)
4760   334 CONTINUE
4770   337 IF (LPP .EQ. 1) GO TO 350
4780       DO 341 M=1,NSG
4790       DO 342 K=1,IX
4800       PLX(K)=XD(K,M)
4810   342 PLY(K)=PSN2(K,M)
4820       IF (M .GT. 1) GO TO 343
4830       CALL TSCALE(PLY,IX,VLOW,VHIGH,.TRUE.,1)
4840       CALL TSCALE(PLX,IX,RLOW,RHIGH,.TRUE.,0)
4850       GO TO 341
4860   343 CALL TSCALE(PLY,IX,VLOW,VHIGH,.FALSE.,1)
4870       CALL TSCALE(PLX,IX,RLOW,RHIGH,.FALSE.,0)
4880   341 CONTINUE
4890       CALL TSETUP(RLO,RHIGH,0.0,VHIGH,1HE,1HG,LBXD,LBY4)
4900       DO 344 M=1,NSG
4910       DO 345 K=1,IX
4920       PLX(K)=XD(K,M)
4930   345 PLY(K)=PSN2(K,M)
4940       IF (M .EQ. NSG) GO TO 346
4950       CALL TPLOT(.FALSE.,ICHR(M),IX,PLX,PLY)
4960       GO TO 344
4970   346 CALL TPLOT(.TRUE.,ICHR(M),IX,PLX,PLY)
4980   344 CONTINUE
4990       GO TO 350
5000   320 IF (LUP .NE. 3) GO TO 350
5010       DO 324 M=1,NSG
5020       DO 322 K=1,IX
5030       PLX(K)=XP(K,M)
5040   322 PLY(K)=PS1(K,M)
5050       IF (M .GT. 1) GO TO 323
5060       CALL TSCALE(PLY,IX,VLOW,VHIGH,.TRUE.,1)
5070       GO TO 324
5080   323 CALL TSCALE(PLY,IX,VLOW,VHIGH,.FALSE.,1)
5090   324 CONTINUE
5100       CALL TSETUP(0.0,XMAX,0.0,VHIGH,1HE,1HG,LBXP,LBYM)
5110       DO 321 M=1,NSG
5120       DO 326 K=1,IX
5130       PLX(K)=XP(K,M)
5140   326 PLY(K)=PS1(K,M)
5150       IF (M .EQ. NSG) GO TO 325
5160       CALL TPLOT(.FALSE.,ICHR(M),IX,PLX,PLY)
5170       GO TO 321
5180   325 CALL TPLOT(.TRUE.,ICHR(M),IX,PLX,PLY)
5190   321 CONTINUE
5200   350 CONTINUE
5210   400 CONTINUE
5220   999 END
@@END
```

```
      SUBROUTINE APPROX
      DIMENSION F(120),KS(60 ),XP(22,5),XD(22,5),PS1(22,5),
     1 PS2(22,5),XDB1(22,5),PSN1(22,5),PSN2(22,5),PSILK(42),
     2 RLK(42),XT(41,5),PAT(41,5)
      DOUBLE PRECISION DELT0,DELT,R,TFINAL,XMSG,XR,DELT1,
     1 DELT2,DELT3,DELT4,DELTI
      COMMON /BLK1/XP,XD,PS1,PS2,XDB1,IX,IFLAG,M,PSN1,PSN2,
     1 RLO,XMAX,ASC,XT,PAT,TMAX
      COMMON N,FRAT,A0R0,SIG0,F,LPS,KS,NHL,NHL21,NFR,NHL2,LUP,
     1 XMSG,IAP,KSN,KSN1,KSN2,AK0A,IBM,RBM
C     IAP=1 FOR 1ST APPROX. FORM, =2 FOR 2ND APPROX. FORM
      IX=0
      IF (LPS .GE. 2) GO TO 41
      IF (LPS .EQ. 1) GO TO 42
      R=0.D0
      RLO=0.0
      TFINAL=XMSG
      DELT0=TFINAL/20.0D0
      DELT1=TFINAL/1.0D4
      DELT2=TFINAL/1.0D3
      DELT3=TFINAL/1.0D2
      DELT4=TFINAL/1.0D1
      GO TO 43
   42 R=1.0D0
      RLO=1.0
      TFINAL=DEXP(XMSG)
      DELT0=DEXP(XMSG/20.0D0)-1.0D0
      DELT1=DEXP(XMSG/1.0D4)-1.0D0
      DELT2=DEXP(XMSG/1.0D3)-1.0D0
      DELT3=DEXP(XMSG/1.0D2)-1.0D0
      DELT4=DEXP(XMSG/1.0D1)-1.0D0
      GO TO 43
   41 R=0.D0
      RLO=0.0
      TFINAL=DEXP(XMSG)/2.0D0
      DELT0=DEXP(XMSG/20.0D0)-1.0D0
      DELT1=DEXP(XMSG/1.0D4)-1.0D0
      DELT2=DEXP(XMSG/1.0D3)-1.0D0
      DELT3=DEXP(XMSG/1.0D2)-1.0D0
      DELT4=DEXP(XMSG/1.0D1)-1.0D0
   43 IF (LUP .NE. 2) GO TO 30
C     PRESTORE PSI1 CURVE
C     THERE ARE OTHER TECHNIQUES FOR EVALUATING THE NESTED INTEGRAL
      XX=0.0
      IF (LPS .EQ. 1) XX=1.0
      DO 100 IR=1,42
      XR=XX
      RLK(IR)=XX
      PSILK(IR)=FPSI(XR,A0R0,SIG0,LPS,IAP)
      IF (PSILK(IR) .LE. 1.0E-20) PSILK(IR)=1.0E-20
      IF (IR .LE. 10) DELTI=DELT1
      IF (IR .GT. 10 .AND. IR .LE. 20) DELTI=DELT2
      IF (IR .GT. 20 .AND. IR .LE. 30) DELTI=DELT3
      IF (IR .GT. 30) DELTI=DELT4
      IF (LPS .EQ. 0) DELX=DELTI
      IF (LPS .EQ. 1) DELX=XX*DELTI
      IF (LPS .GE. 2) DELX=(XX+0.5)*DELTI
      XX=XX+DELX
  100 CONTINUE
   30 IF (IBM .NE. 1) GO TO 20
      R=RBM
      TMX=TMAX*3.14159/180.0
      NTH=41
      GO TO 21
   20 NTH=1
      TMX=0.0
   21 DO 200 ITH=1,NTH
      THET=(ITH-1)*TMX/40.0
      XT(ITH,M)=THET*180.0/3.14159
   40 IX=IX+1
      IF (IBM .NE. 1) XD(IX,M)=R
      IF (LUP .NE. 1) GO TO 50
      PSI1=FPSI(R,A0R0,SIG0,LPS,IAP)
      IF (PSI1 .LE. 1.0E-20) PSI1=1.0E-20
      PS1(IX,M)=PSI1
      IF (LPS .GE. 2) GO TO 52
      IF (LPS .EQ. 1) GO TO 51
      IF (NSC .EQ. 0) XP(IX,M)=R
      IF (NSC .EQ. 1) XP(IX,M)=SIG0*R
      XDB1(IX,M)=-20.0*ALOG10(PSI1)-8.686*A0R0*R
      GO TO 60
   51 IF (NSC .EQ. 0) XP(IX,M)=DLOG(R)
      IF (NSC .EQ. 1) XP(IX,M)=SIG0*DLOG(R)
      XDB1(IX,M)=-20.0*ALOG10(PSI1)-8.686*(DLOG(R)+A0R0*
     1 (R-1.0D0))
      GO TO 60
   52 IF (NSC .EQ. 0) XP(IX,M)=DLOG(R+DSQRT(R**2+1.0D0))
      IF (NSC .EQ. 1) XP(IX,M)=SIG0*DLOG(R+DSQRT(R**2+1.0D0))
      XDB1(IX,M)=-20.0*ALOG10(PSI1)-8.686*(A0R0*R+
     1 DLOG(DSQRT(1.0D0+R**2)))
      GO TO 60
   50 IF (LUP .NE. 2) GO TO 60
      APRR=-R*A0R0*(1.0+FRAT)**2
      IF (LPS .EQ. 0) DPSM=1.0
      IF (LPS .EQ. 1) DPSM=R
      IF (LPS .GE. 2) DPSM=SQRT(1.0+R**2)
      COEF=(SIG0/2.0)*EXP(APRR)/DPSM
      THET2=THET/2.0
      SCK=2.0*FRAT*(AK0A**2)/2.0 *SIN(THET2)**2
      IF (R .LE. 0.D0) GO TO 61
      A=0.0
      IF (LPS .EQ. 1) A=1.0
      B=R
      HSTAR=(B-A)*1.0E-2
      HMIN=(B-A)*1.0E-4
      HMAX=(B-A)/2.0
      ERMAX=1.0E-3
      KEY=0
      CALL ROMBS(A,B,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,KSTOP,KEY)
   62 IF (LPS .EQ. 0) DNI=1.0
      IF (LPS .EQ. 1) DNI=X
      IF (LPS .GE. 2) DNI=SQRT(1.0+X**2)
      IF (X .GT. RLK(1)) GO TO 104
      PSIC=PSILK(1)
      GO TO 103
  104 PM=PSILK(1)
      RM=RLK(1)
      PSIC=1.0E-20
      DO 101 IR=2,42
      IF (X .GT. RLK(IR)) GO TO 102
      RP=RLK(IR)
      PP=PSILK(IR)
      PSIC=((X-RM)/(RP-RM))*(PP-PM)  +PM
      GO TO 103
  102 RM=RLK(IR)
      PM=PSILK(IR)
      IF (IR .LT. 42) GO TO 101
      PSIC=PSILK(42)
      GO TO 103
  101 CONTINUE
  103 CONTINUE
      FOFX=DNI*PSIC*COS(SCK*X)*EXP(A0R0*X)
      CALL ROM2
      IF (KSTOP .EQ. 1) GO TO 62
      GO TO 63
   61 ANS=0.0
   63 PPR=COEF*ANS
      IF (IBM .NE. 1) PSN2(IX,M)=ABS(PPR)
      IF (IBM .NE. 1) GO TO 60
      IF (ITH .EQ. 1) PAT0=ABS(PPR)
      PAT(ITH,M)=20.0*ALOG10(ABS(PPR)/PAT0)
      IF (PAT(ITH,M) .LT. -50.0) PAT(ITH,M)=-50.0
      GO TO 200
   60 CONTINUE
      IF (LPS .EQ. 0) DELT=DELT0
      IF (LPS .EQ. 1) DELT=R*DELT0
      IF (LPS .GE. 2) DELT=(R+0.5D0)*DELT0
      R=R+DELT
      IF (IX .GE. 22) GO TO 200
      IF (R .LE. TFINAL) GO TO 40
  200 CONTINUE
      RETURN
      END
$$END
```

```
      SUBROUTINE DERIV(R,Y)
      DOUBLE PRECISION R,Y(120),R21,RR21,R21I,RI,RLAST,XMSG
      COMMON N,FRAT,A0R0,SIG0,F,LPS,KS,NHL,NHL21,NFR,NHL2,LUP,
     1 XMSG,IAP,KIN,KSN1,KSN2,AK0A,IBM,RBM
      COMMON /BLK2/TDPD,DDT,IA,RLAST,RGOOD,YGOOD,YLAST,IYN,
     1 SWK
      DIMENSION F(120),KS(60 ),RGOOD(60 ),YGOOD(60 ),YLAST(60 ),
     1 IYN(60 )
 1010 FORMAT (1X,4H KI=,I3,2X,3H R=,E10.4)
C     LPS=0 FOR PLANE WAVE,=1 FOR SPHERICAL,=2 FOR COMBINED PHASE,
C        =3 FOR COMBINED NO PHASE
      IF (LPS .GE. 2 .OR. LUP .EQ. 2) GO TO 14
      DO 15 KI=2,N
      IF (Y(2*KI) .GE. 0.D0) GO TO 16
      IF (IYN(KI) .NE. 1) PRINT 1010,KI,R
      IYN(KI)=1
      RGOOD(KI)=RLAST
      YGOOD(KI)=YLAST(KI)
      GO TO 17
   16 YLAST(KI)=Y(2*KI)
   17 IF (IYN(KI) .NE. 1) GO TO 15
      IF (Y(2*KI) .GE. 0.D0) GO TO 15
C     NNN=KS(KI)
C     EARG=-(NNN**2)*A0R0*(R-RGOOD(KI))
C     EXA=EXP(EARG)
C     IF (LPS .EQ. 0) DENR=1.0
C     IF (LPS .EQ. 1) DENR=R/RGOOD(KI)
C     Y(2*KI)=YGOOD(KI)*EXA/DENR
      Y(2*KI)=0.D0
   15 CONTINUE
   14 CONTINUE
      INF2=2*N+1
      SWT=TDPD*DDT*1500.0*AK0A**2 /2.0
      ARF2=A0R0*(FRAT)**2
      FS2=0.25*FRAT*SIG0
      R21=R**2+1.0D0
      RR21=R/R21
      R21I=1.0D0/R21
      IF (LPS .EQ. 1) RI=1.0D0/R
      DO 100 L1=1,N
      SUM1=0.
      SUM2=0.
      KI=KS(LI)
      ATTF=1.0+TDPD/(1.0+KI**2*SWK**2)
      DO 101 MM=1,INF2
      M=MM-N-1
      IF (M .EQ. 0) GO TO 101
      IMA=IABS(M)
      MA=KS(IMA)
      IF (M .LT. 0) MA=-MA
      VM=Y(2*IMA-1)
      WM=Y(2*IMA)
      IF (M .LT. 0) WM=-WM
      NM=KI-MA
      IF (NM .EQ. 0) GO TO 101
      LNMA=IABS(NM)
      IF (LNMA .GT. KS(N)) GO TO 101
      IF (NFR .LE. NHL2) GO TO 11
      MFNM=MOD(LNMA,NFR)
      IF (MFNM .EQ. 0) MFNM=NFR
      MH=IABS(NFR-MFNM)
      IF (MFNM .LE. NHL .OR. MH .LE. NHL) GO TO 10
      GO TO 101
   10 QNMN=(LNMA-0.5)/NFR
      NMDN=INT(QNMN)
      MD=MFNM
      IF (MH .LE. NHL) MD=NHL21-MH
      INMA=NMDN*NHL21+MD
      GO TO 12
   11 INMA=LNMA
   12 CONTINUE
      VNM=Y(2*INMA-1)
      WNM=Y(2*INMA)
      IF (NM .LT. 0) WNM=-WNM
      SUM1=SUM1+   (WNM*VM+VNM*WM)
      SUM2=SUM2+   (WNM*WM-VNM*VM)
  101 CONTINUE
      IF (LUP .NE. 3 .OR. KI .GT. NHL) GO TO 13
      IF (LPS .GE. 2) GO TO 13
      FRN=FRAT*KI
      FRRN=FRN*R
      DFA=(FRN+FRRN**2)/(1.0+FRRN**2)
      DFB=FRRN*(1.0-FRN)/(1.0+FRRN**2)
      TNLN1=KI*FS2*(DFA*SUM1+DFB*SUM2)
      TNLN2=KI*FS2*(DFA*SUM2-DFB*SUM1)
      GO TO 18
   13 TNLN1=KI*FS2*SUM1
      TNLN2=KI*FS2*SUM2
   18 CONTINUE
      TLN1=-Y(2*LI-1)*ARF2*KI**2 *ATTF
      TLN1=TLN1-Y(2*L1)   *KI*SWT/(1.0+KI**2 *SWK**2)
      TLN2=-Y(2*LI)   *ARF2*KI**2 *ATTF
      TLN2=TLN2+Y(2*LI-1)*KI*SWT/(1.0+KI**2 *SWK**2)
      IF (LPS .EQ. 0) GO TO 104
      IF (LPS .EQ. 1) GO TO 105
      IF (LPS .EQ. 2) GO TO 106
      TLN1=TLN1-RR21*Y(2*LI-1)
      TLN2=TLN2-RR21*Y(2*LI)
      GO TO 104
  106 TLN1=TLN1-RR21*Y(2*LI-1)+R21I*Y(2*LI)
      TLN2=TLN2-RR21*Y(2*LI)   -R21I*Y(2*L1-1)
      GO TO 104
  105 TLN1=TLN1-RI*Y(2*LI-1)
      TLN2=TLN2-RI*Y(2*LI)
  104 CONTINUE
      F(2*LI-1)=TLN1+TNLN1
      F(2*LI)   =TLN2+TNLN2
  100 CONTINUE
      RLAST=R
      RETURN
      END
99END
```

```
      SUBROUTINE OUTPUT(R,Y)
      DOUBLE PRECISION R,Y(120),RLN,YMAG,XMSG
      COMMON N,FRAT,A0R0,SIG0,F,LPS,KS,NHL,NHL21,NFR,NHL2,LUP,
     1 XMSG,IAP,KSN,KSN1,KSN2,AK0A,IBM,RBM
      DIMENSION F(120),KS(60 )
      COMMON /BLK1/XP,XD,PS1,PS2,XDB1,IX,IFLAG,M,PSN1,PSN2,
     1 RLD,XMAX,NSC,XT,PAT,TMAX
      DIMENSION XP(22,5),XD(22,5),PS1(22,5),PS2(22,5),XDB1(22,5),
     1 PSN1(22,5),PSN2(22,5),XT(41,5),PAT(41,5)
 2030 FORMAT (7H IFLAG=,I3)
 3010 FORMAT (3H R=,E10.4,3X,8H SIG0*R=,E10.4,3X,7H LN(R)=,E10.4,3X,
     1 /12H SIG0*LN(R)=,E10.4,3X,17H SIG0*ARCSINH(R)=,E10.4)
 3020 FORMAT (3X,2H N,5X,2H V,11X,2H W,11X,4H MAG,9X,9H EXTRA DB,
     1 5H LOSS)
 3030 FORMAT (2X,I3,4(3X,E10.4))
 3040 FORMAT (/6H SIG0=,E10.4)
      IF (IX .EQ. 0) PRINT 3040,SIG0
      PRINT 2030,IFLAG
      SR=R
      IF (NSC .EQ. 1) SR=SIG0*SR
C     LPS=0 FOR PLANE WAVE, =1 FOR SPHERICAL, =2 FOR COMBINED PHASE,
C        =3 FOR COMBINED NO PHASE
      IF (R .LT. 1.0D-20) GO TO 10
      RLN=DLOG(R)
      IF (LPS .EQ. 0) DBL=8.686*A0R0*R
      IF (LPS .EQ. 1) DBL=8.686*(RLN+A0R0*(R-1.0D0))
      IF (LPS .GE. 2) DBL=8.686*(A0R0*R+DLOG(DSQRT(1.0D0+R**2)))
      S0LR=RLN
      IF (NSC .EQ. 1) S0LR=SIG0*S0LR
      GO TO 11
   10 RLN=-9.99D9
      S0LR=-9.99E9
   11 CONTINUE
      S0GR=DLOG(R+DSQRT(R**2+1.0D0))
      IF (NSC .EQ. 1) S0GR=SIG0*S0GR
      IF (IFLAG .EQ. 5) GO TO 16
      IX=IX+1
      IF (LPS .EQ. 0) XP(IX,M)=SR
      IF (LPS .EQ. 1) XP(IX,M)=S0LR
      IF (LPS .GE. 2) XP(IX,M)=S0GR
      XD(IX,M)=R
   16 CONTINUE
```

```
520 C      IF (NSC .EQ. 1) PRINT 3010,R,SR,RLN,SOLR,SUGR
530 C      PRINT 3020
540        DO 301 KI=1,N
550        K=KS(KI)
560        K21=2◆KI-1
570        K2=2◆KI
580        YMAG=DSQRT(Y(K21)◆◆2+Y(K2)◆◆2)
590        IF (YMAG .LT. 1.0D-20 .OR. R .LT. 1.0D-20) GO TO 12
600        EDBL=-20.0◆DLOG10(YMAG)-DBL
610        GO TO 13
620     12 EDBL= 9.99E9
630     13 CONTINUE
640        IF (IFLAG .EQ. 5) GO TO 14
650        IF (KI .NE. 1) GO TO 15
660        PS1(IX,M)=YMAG
670        XDB1(IX,M)=EDBL
680        GO TO 14
690     15 IF (KI .NE. 2) GO TO 17
700        PS2(IX,M)=YMAG
710        GO TO 14
720     17 IF (KI .NE. KSN1) GO TO 18
730        PSN1(IX,M)=YMAG
740        GO TO 14
750     18 IF (KI .NE. KSN2) GO TO 14
760        PSN2(IX,M)=YMAG
770        GO TO 14
780     14 CONTINUE
790        IF (KI .GT. 2) GO TO 301
800 C      PRINT 3030,K,Y(K21),Y(K2),YMAG,EDBL
810    301 CONTINUE
820        RETURN
830        END
99END
```

```
100       FUNCTION FPSI(R,A0R0,SIG0,LPS,IAP)
110       DOUBLE PRECISION R,ARG1,ARG2,ARG,ARGN,DEI,E1T,SQA,DEN,DNR
120       IF (IAP .EQ. 2) GO TO 30
130       ARGN=-A0R0*R
140       ARG1=A0R0*2.0D0
150       ARG=ARG1*2.0D0
160       IF (LPS .GE. 2) GO TO 47
170       IF (LPS .EQ. 1) GO TO 46
180       PSI1=DEXP(ARGN)
190       GO TO 50
200    46 ARG2=ARG1*R
210       DEN=R
220       GO TO 45
230    47 ARG2=ARG1*(1.0D0+R)
240       DEN=1.0D0+R
250    45 E1T=(-DEI(-ARG1))-(-DEI(-ARG2))
260       SQA=1.0D0+(SIG0/2.0D0)**2 *DEXP(ARG)*E1T**2
270       PSI1=DEXP(ARGN)/(DEN*DSQRT(SQA))
280       GO TO 50
290    30 IF (R .LE. 0.D0) GO TO 11
300       A=0.0
310       IF (LPS .EQ. 1) A=1.0
320       B=R
330       HSTAR=(B-A)*1.0E-2
340       HMIN=(B-A)*1.0E-4
350       HMAX=(B-A)/2.0
360       ERMAX=1.0E-3
370       KEY=0
380       CALL ROMBS(A,B,X,FOFX,HSTAR,HMIN,HMAX,ERMAX,ANS,KSTOP,KEY)
390    10 ARGE=-2.0*A0R0*X
400       IF (LPS .EQ. 0) DNI=1.0
410       IF (LPS .EQ. 1) DNI=X
420       IF (LPS .GE. 2) DNI=SQRT(1.0+X**2)
430       FOFX=EXP(ARGE)/DNI
440       CALL ROM2
450       IF (KSTOP .EQ. 1) GO TO 10
460       GO TO 20
470    11 ANS=0.0
480    20 SQA2=1.0+(SIG0/2.0)**2 *ANS**2
490       ARGN=-A0R0*R
500       IF (LPS .EQ. 0) DNR=1.0D0
510       IF (LPS .EQ. 1) DNR=R
520       IF (LPS .GE. 2) DNR=DSQRT(1.0D0+R**2)
530       PSI1=DEXP(ARGN)/(DNR*SQRT(SQA2))
540    50 CONTINUE
550       FPSI=PSI1
560       RETURN
570       END
$$END
```